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The Pedal Curves Generated by Alternative Frame Vectors and Their Smarandache Curves

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Abstract: In this paper, pedal-like curves are defined resulting from the orthogonal projection of a fixed point on the alternative frame vectors of a given regular curve. For each pedal curve, the Frenet vectors, the curvature and the torsion functions are found to provide the common relations among the main curve and its pedal curves. Then, Smarandache curves are defined by using the alternative frame vectors of each pedal curve as position vectors. The relations of the Frenet apparatus are also established for the pedal curves and their corresponding Smarandache curves. Finally, the expressions of the alternative frame apparatus of each Smarandache curves are given in terms of the alternative frame elements of the pedal curves. Thus, a set of new symmetric curves are introduced that contribute to the vast curve family.

Keywords: pedal curve; alternative frame; Smarandache curve

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1. Introduction

Curves are natural figures that always exist in nature. Scientists have formulated these figures, examined their characteristics and obtained some theories. Curve theory has been one of the fields of study that best represents differential geometry and makes it most interesting. More information and theory can be obtained when an orthonormal system is established on a curve. Following such an orthonormal system, the curvatures which are the invariants of a curve can be established. Thus, one can characterize a given curve by these invariants. For example, if the second curvature (torsion) of a curve is zero, then the curve is a planar curve; if the torsion is different from zero, then the curve is a space curve. If the harmonic curvature, which is the ratio of the torsion to curvature, is constant, then the curve is a general helix. If the curvature is a constant and the torsion is a function, the curve is a Salkowski curve [1,2].

Another useful aspect of curve theory is to associate some curves to each other. Any two curves can be related to one another by some specific and symmetrical methods. Such methods of symmetry are based on translations, scaling, rotations and reflections. For example, a Bertrand pair has common principal normal vectors, that is to say, the principal normal vector of a Bertrand curve is translated and its mate curve accepts this translated principal normal vector. As a consequence, the resulting Bertrand pair has the relation of scale symmetry (see Figure 1). Similarly, a Mannheim curve pair possesses such translational symmetry between the principal normal and binormal vectors, whereas a successor curve has the translational symmetry between the tangent vector and principal normal vector. However, the involute–evolute curves have a 90-degree rotary symmetry relation between their tangent vectors [3]. There are many studies on these curves; see [4] and the references therein.

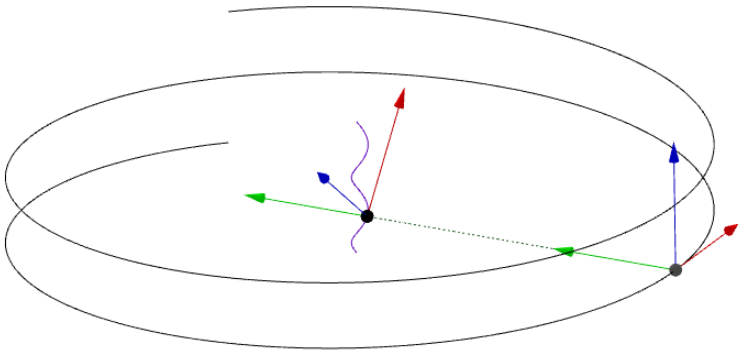

Figure 1. A helix curve (purple) and its Bertrand mate (black). The principal normal vectors (green) have a translational symmetry, whereas the resulting pair of curves have a scale symmetry relation between them. To see the animated version of this figure, please visit https://www.geogebra.org/m/ndjktahz, accessed on 9 July 2024.

Further, generating new curves is an important matter for the theory of curves. Smarandache geometry is one way of creating new curves [5–7]. In addition, a pedal curve is another type of curve which is derived by a given specific curve and a fixed point. That is, a pedal (resp., contrapedal) curve is defined as the geometric location of the projection points on the tangent (resp., normal vector) of a curve from a point that is not on the curve. This way of deriving new curves is also considered as a symmetric procedure; see [3]. Many studies exist on these curves such as [8–10] and the references therein. Further, research on such curves has been carried out using different frames in many spaces, and similar works continue today [11–16].

In this study, pedal curves belonging to the alternative frame vectors of a space curve are defined, and their Frenet apparatus is calculated. Next, Smarandache curves are defined by taking alternative frame elements of each pedal curve as the position vectors. Finally, the corresponding alternative frame apparatus are obtained and expressed in terms of the main curve. In the following lines, we recall the basic concepts that we use throughout the paper. For a given differentiable curve $\alpha(t) : I \subset R \to R^3$, the formulae of Frenet vector fields and curvature functions are defined as follows:

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad N = B \wedge T = \frac{(\alpha' \wedge \alpha'') \wedge \alpha'}{\|\alpha' \wedge \alpha''\| \|\alpha'\|}, \quad B = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \tag{1}$$

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \quad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}, \tag{2}$$

$$T' = v\kappa N, \quad N' = v(-\kappa T + \tau B), \quad B' = -v\tau N, \tag{3}$$

where $\| \ \|$ is the norm, "$\wedge$" is the vector product operator, and $v = \|\alpha'\|$ [10,17]. Frenet vectors make a rotational motion along the curve around an axis. This axis is called the Darboux rotational axis, and it is defined by the relation $W_0 = \tau T + \kappa B$. If the unit Darboux vector is denoted by W, then, for $\phi = \measuredangle(B, W_0)$, we have the following equation.

$$W = \sin\phi T + \cos\phi B, \tag{4}$$

where $\cos\phi = \dfrac{\kappa}{\sqrt{\kappa^2 + \tau^2}}$ and $\sin\phi = \dfrac{\tau}{\sqrt{\kappa^2 + \tau^2}}$. Since W is a vector lying in the rectifying plane, it is clear that $N \perp W$. Thus, a new orthonormal system on a curve can be established with the set of vectors $\{N, C = N \wedge W, W\}$. The new moving frame is called the alterna-

tive frame for the curve. We have the following relations among Frenet and alternative frame vectors.

$$N = N, \qquad C = -\cos\phi T + \sin\phi B, \qquad W = \sin\phi T + \cos\phi B. \tag{5}$$

Further, similar to the Frenet formulae given in (3), we have the alternative formulae as follows:

$$N' = v\beta C, \qquad C' = v(-\beta N + \phi' W), \qquad W' = -v\phi' C, \tag{6}$$

where $\phi' = \left(\frac{\tau}{\kappa}\right)'\left(\frac{\kappa^2+\tau^2}{\kappa^2}\right)$, and $\beta = \kappa\sqrt{\kappa^2+\tau^2}$.

2. The Alternative N-Pedal Curve and Its Smarandache Curves

In this section of the study, we first define the alternative N-pedal curve considering the principal normal vector of a given curve as an element of the alternative frame. Second, we find the Frenet apparatus of the alternative N-pedal curve and establish the relationship among the alternative frame vectors of both the main curve and its corresponding alternative N-pedal curve. Then, we examine four possible Smarandache curves of the alternative N-pedal curve whose position vectors are the linear combinations of the alternative frame elements. We also support the results by giving an example and the graphs of the curves.

Definition 1. *Let $\alpha : I \subset R \to R^3$ be a regular curve in E^3, and $\{N, C, W\}$ denote the set of its alternative frame vectors. Then, the locus of the perpendicular projection points of a fixed point P that is not on α onto the alternative frame vector N is called the alternative N-pedal curve for α according to P (see Figure 2).*

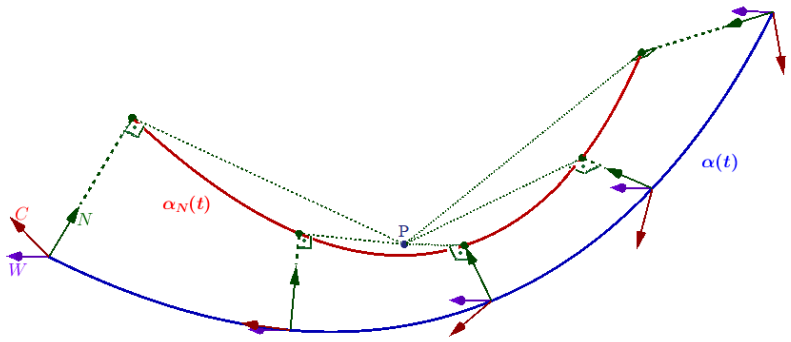

Figure 2. The alternative N-pedal curve (red) of the curve $\alpha(t)$ (blue) according to a fixed point P.

Theorem 1. *The equation of the alternative N-pedal curve of the curve α is given as follows:*

$$\alpha_N(t) = \alpha(t) + \langle P - \alpha(t), N(t)\rangle N(t). \tag{7}$$

Proof. Let P' be the perpendicular projection point onto the normal vector $\overrightarrow{N}$ from the fixed point P that is not on the curve α. By using the definition of vector projections, we compute the perpendicular projection vector denoted by $\overrightarrow{\alpha P'}$ as follows:

$$\overrightarrow{\alpha P'} = \frac{\left\langle \overrightarrow{\alpha P}, \overrightarrow{N} \right\rangle}{\left\langle \overrightarrow{N}, \overrightarrow{N} \right\rangle}\overrightarrow{N}.$$

Thus, with the motion of the frame vectors along the curve, the geometric location of the point P' constructs a new curve that is called the alternative N-pedal curve and denoted by $\alpha_N(t)$. Accordingly, we have the following equations:

$$\begin{aligned}\overrightarrow{\alpha P} = \overrightarrow{\alpha P'} + \overrightarrow{P'P} \Rightarrow \overrightarrow{\alpha P} &= \frac{\left\langle \overrightarrow{\alpha P}, \overrightarrow{N} \right\rangle}{\left\| \overrightarrow{N} \right\|^2} \overrightarrow{N} + \overrightarrow{P'P} \\ &\Rightarrow P' = \alpha(t) + \left\langle \overrightarrow{\alpha P}, \overrightarrow{N} \right\rangle \overrightarrow{N} \\ &\Rightarrow \alpha_N(t) = \alpha(t) + \langle P - \alpha(t), N(t) \rangle N(t),\end{aligned}$$

by which the proof is completed. □

If specifically $\langle P - \alpha(t), N(t) \rangle = u(t)$, then Relation (7) can clearly be simplified as follows:

$$\alpha_N(t) = \alpha(t) + u(t)N(t). \tag{8}$$

Theorem 2. *Let α_N be the alternative N-pedal curve of a unit speed curve α, and $\{T_1,\ N_1,\ B_1\}$ denote the Frenet vectors of the alternative N-pedal curve. Then, we have the following relations:*

$$\begin{aligned}T_1 =& \omega_1 u' N + \omega_1 (uv\beta - \cos\varphi) C + \omega_1 \sin\varphi W, \\ N_1 =& B_1 \wedge T_1, \\ B_1 =& \eta_1 (x_3 uv\beta - x_3 \cos\varphi - x_2 \sin\varphi) N + \eta_1 \left(x_1 \sin\varphi - u' x_3\right) C \\ &+ \eta_1 \left(u' x_2 - x_1 uv\beta + x_1 \cos\varphi\right) W,\end{aligned}$$

where

$$\begin{aligned}x_1 =& u'' - v^2\beta(u\beta - \cos\varphi), \\ x_2 =& v\left(u'\beta - v\varphi' \sin\varphi\right) + (uv\beta - v\cos\varphi)', \\ x_3 =& v^2\varphi'(u\beta - \cos\varphi) + (v\sin\varphi)', \\ \omega_1 =& \frac{1}{\sqrt{(u')^2 + (uv\beta - v\cos\varphi)^2 + (v\sin\varphi)^2}}, \\ \eta_1 =& \frac{1}{\sqrt{\left(x_3 uv\beta - x_3 v\cos\varphi - x_2 v\sin\varphi\right)^2 + \left(x_1 v\sin\varphi - u' x_3\right)^2 + \left(u' x_2 - x_1 uv\beta + x_1 v\cos\varphi\right)^2}}.\end{aligned}$$

Proof. By considering the relations in (5), we have the derivatives up to the third degree for Relation (8) as follows:

$$\begin{aligned}\alpha_N' =& u' N + v(u\beta - \cos\varphi) C + v\sin\varphi W, \\ \alpha_N'' =& x_1 N + x_2 C + x_3 W, \\ \alpha_N''' =& \left(x_1' - x_2 v\beta\right) N + \left(x_2' + x_1 v\beta - x_3 v\varphi'\right) C + \left(x_2 v\varphi' + x_3'\right) W.\end{aligned} \tag{9}$$

Further, if we perform the required vector product operations and take the norms, the following relations are obtained:

$$\begin{aligned}\alpha_N' \wedge \alpha_N'' =& (ux_3 v\beta - x_2 v\sin\varphi - x_3 v\cos\varphi) N + \left(x_1 v\sin\varphi - x_3 u'\right) C \\ &+ \left(x_2 u' - ux_1 v\beta + x_1 v\cos\varphi\right) W,\end{aligned}$$

$$\begin{aligned}\det\left(\alpha_N', \alpha_N'', \alpha_N'''\right) =& \left(-x_2 v\sin\varphi + ux_3 v\beta - x_3 v\cos\varphi\right)\left(x_1' - x_2 v\beta\right) \\ &+ \left(x_1\sin\varphi - u' x_3\right)\left(x_1 v\beta + x_2' - x_3 v\varphi'\right) \\ &+ \left(x_2 u' - ux_1 v\beta + x_1 v\cos\varphi\right)\left(x_2 v\varphi' + x_3'\right),\end{aligned} \tag{10}$$

$$\begin{aligned}\left\|\alpha_N'\right\| =&\sqrt{u'^2+v^2+v^2u\beta(u\beta-\cos\varphi)},\\ \left\|\alpha_N'\wedge\alpha_N''\right\| =&\sqrt{\begin{array}{l}(ux_3v\beta-x_2v\sin\varphi-x_3v\cos\varphi)^2+(x_1v\sin\varphi-x_3u')^2\\ +(u'x_2-x_1uv\beta+x_1v\cos\varphi)^2\end{array}}.\end{aligned}\tag{11}$$

If Equations (9)–(11) are substituted in (1), the proof is completed. □

Theorem 3. *Let α_N be the alternative N-pedal curve of the unit speed curve α, and κ_1 and τ_1 denote its curvature and torsion functions, respectively. Then, we have the following relations among the curvatures:*

$$\begin{aligned}\kappa_1=&\frac{\sqrt{(x_3uv\beta-x_3\cos\varphi-x_2\sin\varphi)^2+(x_1\sin\varphi-u'x_3)^2+(u'x_2-x_1uv\beta+x_1\cos\varphi)^2}}{\left(u'^2+v^2+v^2u\beta(u\beta-\cos\varphi)\right)\sqrt{u'^2+v^2+v^2u\beta(u\beta-\cos\varphi)}},\\ \tau_1=&\frac{\begin{array}{l}(-x_2v\sin\varphi+ux_3v\beta-x_3v\cos\varphi)(x_1'-x_2v\beta)+(x_1\sin\varphi-u'x_3)(x_1v\beta+x_2'-x_3v\varphi')\\ +(x_2u'-ux_1v\beta+x_1v\cos\varphi)(x_2v\varphi'+x_3')\end{array}}{(ux_3v\beta-x_2v\sin\varphi-x_3v\cos\varphi)^2+(x_1v\sin\varphi-x_3u')^2+(u'x_2-x_1uv\beta+x_1v\cos\varphi)^2}.\end{aligned}$$

Proof. We substitute the relations given in (10) and (11) in (2) and complete the proof. □

Remark 1. *The alternative frame for the alternative N-pedal curve α_N is established by the following equations:*

$$N_1=N_1,\qquad C_1=-\cos\partial T_1+\sin\partial B_1,\qquad W_1=\sin\partial T_1+\cos\partial B_1,\tag{12}$$

where $\partial=\measuredangle(B_1,W_1)$, $\cos\partial=\dfrac{\kappa_1}{\sqrt{\kappa_1^2+\tau_1^2}}$, *and* $\sin\partial=\dfrac{\tau_1}{\sqrt{\kappa_1^2+\tau_1^2}}$.

Corollary 1. *The alternative frame formulae for the alternative N-pedal curve is given as follows:*

$$N_1'=\mu_1\beta_1C_1,\qquad C'_1=-\mu_1\beta_1N_1+\mu_1\partial'W_1,\qquad W_1{}'=-\mu_1\partial'C_1,\tag{13}$$

where $\mu_1=\left\|\alpha_N'\right\|$, $\partial'=\left(\dfrac{\tau_1}{\kappa_1}\right)'\left(\dfrac{\kappa_1^2+\tau_1^2}{\kappa_1^2}\right)$, $\beta_1=\mu_1\sqrt{\kappa_1^2+\tau_1^2}$.

Definition 2. *The N_1C_1 Smarandache curve of the alternative N-pedal curve whose position vector is the linear combination of the two vectors N_1 and C_1 is defined by the following parametrization:*

$$\alpha_1=\frac{N_1+C_1}{\sqrt{2}}.\tag{14}$$

Theorem 4. *Let $\{N_{\alpha_1},\ C_{\alpha_1},\ W_{\alpha_1}\}$ denote the alternative frame for the N_1C_1 Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$\begin{aligned}T_{\alpha_1}=&\frac{\sqrt{2}(-\beta_1N_1+\beta_1C_1+\partial'W_1)}{\sqrt{2\beta_1^2+\partial'^2}},\\ N_{\alpha_1}=&B_{\alpha_1}\wedge T_{\alpha_1},\\ B_{\alpha_1}=&\frac{(x_6\beta_1-x_5\partial')N_1+(x_6\beta_1+x_4\partial')C_1-(\beta_1x_4+\beta_1x_5)W_1}{\sqrt{(x_6\beta_1-x_5\partial')^2+(x_6\beta_1+x_4\partial')^2+(\beta_1x_4+\beta_1x_5)^2}},\end{aligned}$$

where $x_4=-\dfrac{(\mu_1\beta_1)'+\mu_1^2\beta_1^2}{\sqrt{2}}$, $x_5=\dfrac{(\mu_1\beta_1)'-\mu_1^2\left(\beta_1^2+\partial'^2\right)}{\sqrt{2}}$, $x_6=\dfrac{(\mu_1\partial')'+\mu_1^2\beta_1\partial'}{\sqrt{2}}$.

Proof. If Relation (13) is taken into account, and the required derivatives for the N_1C_1 curve given in (14) are taken, we have

$$\begin{aligned}\alpha_1' =& \frac{-\mu_1\beta_1 N_1 + \mu_1\beta_1 C_1 + \mu_1\partial' W_1}{\sqrt{2}},\\ \alpha_1'' =& x_4 N_1 + x_5 C_1 + x_6 W_1,\\ \alpha_1''' =& \left(x_4' - x_5\mu_1\beta_1\right)N_1 + \left(x_5' + x_4\mu_1\beta_1 - x_6\mu_1\partial'\right)C_1 + \left(x_6' + x_5\mu_1\partial'\right)W_1.\end{aligned} \tag{15}$$

Moreover, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}\alpha_1' \wedge \alpha_1'' =& \frac{\mu_1}{\sqrt{2}}\left(\left(x_6\beta_1 - x_5\partial'\right)N_1 + \left(x_6\beta_1 + x_4\partial'\right)C_1 - \beta_1(x_4 + x_5)W_1\right),\\ \det\left(\alpha_1', \alpha_1'', \alpha_1'''\right) =& \frac{\mu_1}{\sqrt{2}}\begin{pmatrix} \left(x_4' - x_5\mu_1\beta_1\right)\left(x_6\beta_1 - x_5\partial'\right)\\ +\left(x_5' + x_4\mu_1\beta_1 - x_6\mu_1\partial'\right)\left(x_6\beta_1 + x_4\partial'\right)\\ -\beta_1(x_4 + x_5)\left(x_6' + x_5\mu_1\partial'\right)\end{pmatrix},\end{aligned} \tag{16}$$

and

$$\begin{aligned}\left\|\alpha_1'\right\| =& \frac{\mu_1}{\sqrt{2}}\sqrt{2\beta_1^2 + \partial'^2},\\ \left\|\alpha_1' \wedge \alpha_1''\right\| =& \frac{\mu_1}{\sqrt{2}}\sqrt{\left(x_6\beta_1 - x_5\partial'\right)^2 + \left(x_6\beta_1 + x_4\partial'\right)^2 + \left(x_4\beta_1 + x_5\beta_1\right)^2}.\end{aligned} \tag{17}$$

Finally, by substituting Relations (15)–(17) into (1), we complete the proof. □

Theorem 5. *Let κ_{α_1} and τ_{α_1} be the curvature and the torsion functions for the N_1C_1 Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\begin{aligned}\kappa_{\alpha_1} =& \frac{2\sqrt{\left(x_6\beta_1 - x_5\partial'\right)^2 + \left(x_6\beta_1 + x_4\partial'\right)^2 + \left(\beta_1 x_4 + \beta_1 x_5\right)^2}}{\mu_1^2\left(2\beta_1^2 + \partial'^2\right)\sqrt{2\beta_1^2 + \partial'^2}},\\ \tau_{\alpha_1} =& \frac{\sqrt{2}\begin{pmatrix}\left(x_4' - x_5\mu_1\beta_1\right)\left(x_6\beta_1 - x_5\partial'\right) + \left(x_5' + x_4\mu_1\beta_1 - x_6\mu_1\partial'\right)\left(x_6\beta_1 + x_4\partial'\right)\\ -\beta_1(x_4 + x_5)\left(x_6' + x_5\mu_1\partial'\right)\end{pmatrix}}{\mu_1\left(\left(x_6\beta_1 - x_5\partial'\right)^2 + \left(x_6\beta_1 + x_4\partial'\right)^2 + \left(\beta_1 x_4 + \beta_1 x_5\right)^2\right)}.\end{aligned}$$

Proof. By using (16) and (17) and substituting them into (2), we complete the proof. □

Remark 2. *The alternative frame for the N_1C_1 Smarandache curve α_1 is established by the following equations:*

$$N_{\alpha_1} = N_{\alpha_1},\ C_{\alpha_1} = -\cos\partial_1 T_{\alpha_1} + \sin\partial_1 B_{\alpha_1},\ W_{\alpha_1} = \sin\partial_1 T_{\alpha_1} + \cos\partial_1 B_{\alpha_1},$$

where $\partial_1 = \measuredangle(B_{\alpha_1}, W_{\alpha_1})$, $\cos\partial_1 = \dfrac{\kappa_{\alpha_1}}{\sqrt{\kappa_{\alpha_1}^2 + \tau_{\alpha_1}^2}}$, *and* $\sin\partial_1 = \dfrac{\tau_{\alpha_1}}{\sqrt{\kappa_{\alpha_1}^2 + \tau_{\alpha_1}^2}}$.

Definition 3. *The N_1W_1 Smarandache curve of the alternative N-pedal curve whose position vector is the linear combination of the two vectors N_1 and W_1 is defined by the following parametrization:*

$$\alpha_2 = \frac{N_1 + W_1}{\sqrt{2}}. \tag{18}$$

Theorem 6. *Let $\{N_{\alpha_2}, C_{\alpha_2}, W_{\alpha_2}\}$ denote the alternative frame for the N_1W_1 Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$T_{\alpha_2} = C_1, \qquad N_{\alpha_2} = \frac{-\beta_1 N_1 + \partial' W_1}{\sqrt{\partial'^2 + \beta_1^2}}, \qquad B_{\alpha_2} = \frac{\partial' N_1 + \beta_1 W_1}{\sqrt{\partial'^2 + \beta_1^2}},$$

where ∂' *and* β_1 *are as given in Corollary 1.*

Proof. If Relation (13) is taken into account, and the required derivatives for the N_1W_1 curve given in (18) are taken, we have

$$\begin{aligned}
\alpha_2' =& \frac{\mu_1(\beta_1 - \partial')C_1}{\sqrt{2}}, \\
\alpha_2'' =& \frac{-\mu_1^2\beta_1(\beta - \partial')N_1 + (\mu_1\beta_1 - \mu_1\partial')'C_1 + \mu_1^2\partial'(\beta_1 - \partial')W_1}{\sqrt{2}}, \\
& -\Big((\mu_1^2\beta_1(\beta_1 - \partial'))' + \mu_1\beta_1(\mu_1\beta_1 - \mu_1\partial')'\Big)N_1 \\
& +\Big((\mu_1\beta_1 - \mu_1\partial')'' - \mu_1^3(\beta_1 - \partial')\Big(\beta_1^2 + (\partial_1')^2\Big)\Big)C_1 \\
\alpha_2''' =& \frac{+\Big((\mu_1^2\partial'(\beta_1 - \partial'))' + \mu_1\partial'(\mu_1\beta_1 - \mu_1\partial')'\Big)W_1}{\sqrt{2}}.
\end{aligned} \tag{19}$$

In addition, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}
\alpha_2' \wedge \alpha_2'' =& \frac{\mu_1^3\partial'(\beta_1 - \partial')^2 N_1 + \mu_1^3\beta_1(\beta_1 - \partial')^2 W_1}{2}, \\
\det(\alpha_2', \alpha_2'', \alpha_2''') =& \frac{\mu_1^5(\beta_1 - \partial')^3(\beta_1\partial'' - \partial'\beta_1')}{2\sqrt{2}},
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\|\alpha_2'\| =& \frac{\mu_1(\beta_1 - \partial')}{\sqrt{2}}, \\
\|\alpha_2' \wedge \alpha_2''\| =& \frac{(\beta_1 - \partial')^2\mu_1^3\sqrt{\partial'^2 + \beta_1^2}}{2}.
\end{aligned} \tag{21}$$

Finally, by substituting Relations (19)–(21) into (1), we complete the proof. □

Theorem 7. *Let* κ_{α_2} *and* τ_{α_2} *be the curvature and the torsion functions for the* N_1W_1 *Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\kappa_{\alpha_2} = \frac{\sqrt{2}\sqrt{\partial'^2 + \beta_1^2}}{\beta_1 - \partial'}, \qquad \tau_{\alpha_2} = \frac{\sqrt{2}(\beta_1\partial'' - \partial'\beta_1')}{\mu_1(\beta_1 - \partial')\Big(\partial'^2 + \beta_1^2\Big)}$$

Proof. By using (20) and (21) and substituting them into (2), we complete the proof. □

Remark 3. *The alternative frame for the* N_1W_1 *Smarandache curve* α_2 *is established by the following equations:*

$$N_{\alpha_2} = N_{\alpha_2}, \quad C_{\alpha_2} = -\cos\partial_2 T_{\alpha_2} + \sin\partial_2 B_{\alpha_2}, \quad W_{\alpha_2} = \sin\partial_2 T_{\alpha_2} + \cos\partial_2 B_{\alpha_2}$$

where $\partial_2 = \measuredangle(B_{\alpha_2}, W_{\alpha_2})$, $\cos\partial_2 = \dfrac{\kappa_{\alpha_2}}{\sqrt{\kappa_{\alpha_2}^2 + \tau_{\alpha_2}^2}}$, *and* $\sin\partial_2 = \dfrac{\tau_{\alpha_2}}{\sqrt{\kappa_{\alpha_2}^2 + \tau_{\alpha_2}^2}}$.

Definition 4. *The C_1W_1 Smarandache curve of the alternative N-pedal curve whose position vector is the linear combination of the two vectors C_1 and W_1 is defined by the following parametrization:*

$$\alpha_3 = \frac{C_1 + W_1}{\sqrt{2}}. \tag{22}$$

Theorem 8. *Let $\{N_{\alpha_3}, C_{\alpha_3}, W_{\alpha_3}\}$ denote the alternative frame for the C_1W_1 Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$\begin{aligned} T_{\alpha_3} =& \frac{-\beta_1 N_1 - \partial' C_1 + \partial' W_1}{\sqrt{\beta_1^2 + 2\partial'^2}}, \\ N_{\alpha_3} =& B_{\alpha_3} \wedge T_{\alpha_3}, \\ B_{\alpha_3} =& \frac{-(\partial' x_9 + \partial' x_8)N_1 + (\beta_1 x_9 + \partial' x_7)C_1 + (\partial' x_7 - \beta_1 x_8)W_1}{\sqrt{(\partial' x_9 + \partial' x_8)^2 + (\beta_1 x_9 + \partial' x_7)^2 + (\partial' x_7 - \beta_1 x_8)^2}}, \end{aligned}$$

where $x_7 = \dfrac{\mu_1^2 \partial' \beta_1 - (\mu_1 \beta_1)'}{\sqrt{2}}$, $x_8 = -\dfrac{(\mu_1 \partial')' + \mu_1^2\left(\beta_1^2 + \partial'^2\right)}{\sqrt{2}}$, $x_9 = \dfrac{(\mu_1 \partial')' - \mu_1^2 \partial'^2}{\sqrt{2}}$.

Proof. If Relation (13) is taken into account, and the required derivatives for the C_1W_1 curve given in (22) are taken, we have

$$\begin{aligned} \alpha_3' =& \frac{-\mu_1\beta_1 N_1 - \mu_1 \partial' C_1 + \mu_1 \partial' W_1}{\sqrt{2}}, \\ \alpha_3'' =& x_7 N_1 + x_8 C_1 + x_9 W_1, \\ \alpha_3''' =& (x_7' - x_8 \mu_1 \beta_1) N_1 + (x_8' + x_7 \mu_1 \beta_1 - x_9 \mu_1 \partial') C_1 + (x_9' + x_8 \mu_1 \partial') W_1. \end{aligned} \tag{23}$$

In addition, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned} \alpha_3' \wedge \alpha_3'' =& \frac{\mu_1(-(\partial' x_9 + \partial' x_8)N_1 + (\beta_1 x_9 + \partial' x_7)C_1 + (\partial' x_7 - \beta_1 x_8)W_1)}{\sqrt{2}}, \\ \det(\alpha_3', \alpha_3'', \alpha_3''') =& \frac{\mu_1}{\sqrt{2}} \begin{pmatrix} (\partial' x_7 - \beta_1 x_8)(x_9' + x_8 \mu_1 \partial') - (\partial' x_9 + \partial' x_8)(x_7' - x_8 \mu_1 \beta_1) \\ +(\beta_1 x_9 + \partial' x_7)(x_8' + x_7 \mu_1 \beta_1 - x_9 \mu_1 \partial') \end{pmatrix}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} \left\|\alpha_3'\right\| =& \frac{\mu_1}{\sqrt{2}} \sqrt{\beta_1^2 + 2\partial'^2}, \\ \left\|\alpha_3' \wedge \alpha_3''\right\| =& \frac{\mu_1}{\sqrt{2}} \sqrt{(\partial' x_9 + \partial' x_8)^2 + (\beta_1 x_9 + \partial' x_7)^2 + (\partial' x_7 - \beta_1 x_8)^2}. \end{aligned} \tag{25}$$

Finally, by substituting Relations (23)–(25) into (1), we complete the proof. □

Theorem 9. *Let κ_{α_3} and τ_{α_3} be the curvature and the torsion functions for the C_1W_1 Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\begin{aligned} \kappa_{\alpha_3} =& \frac{2\sqrt{(\partial' x_9 + \partial' x_8)^2 + (\beta_1 x_9 + \partial' x_7)^2 + (\partial' x_7 - \beta_1 x_8)^2}}{\mu_1^2\left(\beta_1^2 + 2\partial'^2\right)\sqrt{\beta_1^2 + 2\partial'^2}}, \\ \tau_{\alpha_3} =& \frac{\sqrt{2}\begin{pmatrix} (\partial' x_7 - \beta_1 x_8)(x_9' + x_8 \mu_1 \partial') - (\partial' x_9 + \partial' x_8)(x_7' - x_8 \mu_1 \beta_1) \\ +(\beta_1 x_9 + \partial' x_7)(x_8' + x_7 \mu_1 \beta_1 - x_9 \mu_1 \partial') \end{pmatrix}}{\mu_1\left((\partial' x_9 + \partial' x_8)^2 + (\beta_1 x_9 + \partial' x_7)^2 + (\partial' x_7 - \beta_1 x_8)^2\right)}. \end{aligned}$$

Proof. By using (24) and (25) and substituting them into (2), we complete the proof. □

Remark 4. *The alternative frame for the C_1W_1 Smarandache curve α_3 is established by the following equations:*

$$N_{\alpha_3} = N_{\alpha_3}, \qquad C_{\alpha_3} = -\cos\partial_3 T_{\alpha_3} + \sin\partial_3 B_{\alpha_3}, \qquad W_{\alpha_3} = \sin\partial_3 T_{\alpha_3} + \cos\partial_3 B_{\alpha_3}$$

where $\partial_3 = \measuredangle(B_{\alpha_3}, W_{\alpha_3})$, $\cos\partial_3 = \dfrac{\kappa_{\alpha_3}}{\sqrt{\kappa_{\alpha_3}^2 + \tau_{\alpha_3}^2}}$, *and* $\sin\partial_3 = \dfrac{\tau_{\alpha_3}}{\sqrt{\kappa_{\alpha_3}^2 + \tau_{\alpha_3}^2}}$.

Definition 5. *The $N_1C_1W_1$ Smarandache curve of the alternative N-pedal curve whose position vector is the linear combination of the vectors N_1, C_1 and W_1 is defined by the following parametrization:*

$$\alpha_4 = \frac{N_1 + C_1 + W_1}{\sqrt{3}}. \tag{26}$$

Theorem 10. *Let $\{N_{\alpha_4}, C_{\alpha_4}, W_{\alpha_4}\}$ denote the alternative frame for the $N_1C_1W_1$ Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$\begin{aligned}
T_{\alpha_4} =& \frac{-\beta_1 N_1 + (\beta_1 - \partial')C_1 + \partial' W_1}{\sqrt{2}\sqrt{\beta_1^2 - \beta_1\partial' + \partial'^2}},\\
N_{\alpha_4} =& B_{\alpha_4} \wedge T_{\alpha_4},\\
B_{\alpha_4} =& \frac{-(x_{11}\partial' - x_{12}(\beta_1 - \partial'))N_1 + (x_{10}\partial' + x_{12}\beta_1)C_1 - (x_{11}\beta_1 + x_{10}(\beta_1 - \partial'))W_1}{\sqrt{(x_{11}\partial' - x_{12}(\beta_1 - \partial'))^2 + (x_{10}\partial' + x_{12}\beta_1)^2 + (x_{11}\beta_1 + x_{10}(\beta_1 - \partial'))^2}},
\end{aligned}$$

where $x_{10} = \dfrac{(\mu_1\beta_1)' + \mu_1^2\beta_1(\beta_1 - \partial')}{\sqrt{3}}$, $x_{11} = \dfrac{(\mu_1(\beta_1 - \partial'))' - \mu_1^2\left(\beta_1^2 + (\partial_1')^2\right)}{\sqrt{3}}$,
$x_{12} = \dfrac{(\mu_1\partial')' + \mu_1^2\partial'(\beta_1 - \partial')}{\sqrt{3}}$.

Proof. If Relation (13) is taken into account, and the required derivatives for the $N_1C_1W_1$ curve given in (22) are taken, we have

$$\begin{aligned}
\alpha_4' =& \frac{\mu_1(-\beta_1 N_1 + (\beta_1 - \partial')C_1 + \partial' W_1)}{\sqrt{3}},\\
\alpha_4'' =& x_{10}N_1 + x_{11}C_1 + x_{12}W_1,\\
\alpha_4''' =& (x_{10} - x_{11}\mu_1\beta_1)N_1 + \left(x_{11} + x_{10}\mu_1\beta_1 - x_{12}\mu_1\partial'\right)C_1 + \left(x_{12} + x_{11}\mu_1\partial'\right)W_1.
\end{aligned} \tag{27}$$

In addition, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}
\alpha_4' \wedge \alpha_4'' =& \frac{\mu_1\begin{pmatrix} -(x_{11}\partial' - x_{12}(\beta_1 - \partial'))N_1 + (x_{10}\partial' + x_{12}\beta_1)C_1 \\ -(x_{11}\beta_1 + x_{10}(\beta_1 - \partial'))W_1 \end{pmatrix}}{\sqrt{3}},\\
\det(\alpha_4', \alpha_4'', \alpha_4''') =& \frac{\mu_1}{\sqrt{3}}\begin{pmatrix} -(x_{11}\partial' - x_{12}(\beta_1 - \partial'))(x_{10} - x_{11}\mu_1\beta_1) \\ +(x_{10}\partial' + x_{12}\beta_1)(x_{11} + x_{10}\mu_1\beta_1 - x_{12}\mu_1\partial') \\ -(x_{11}\beta_1 + x_{10}(\beta_1 - \partial'))(x_{12} + x_{11}\mu_1\partial') \end{pmatrix},
\end{aligned} \tag{28}$$

and

$$\begin{aligned}\left\|\alpha_4'\right\| =& \frac{\sqrt{6}\mu_1}{3}\sqrt{\beta_1^2-\beta_1\partial'+\partial'^2},\\ \left\|\alpha_4'\wedge\alpha_4''\right\| =& \frac{\mu_1}{\sqrt{3}}\sqrt{\begin{array}{l}\left(x_{11}\partial'-x_{12}(\beta_1-\partial')\right)^2+\left(x_{10}\partial'+x_{12}\beta_1\right)^2\\ +\left(x_{11}\beta_1+x_{10}(\beta_1-\partial')\right)^2\end{array}}.\end{aligned} \tag{29}$$

Finally, by substituting Relations (27)–(29) into (1), we complete the proof. □

Theorem 11. *Let κ_{α_4} and τ_{α_4} be the curvature and the torsion functions for the $N_1C_1W_1$ Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\kappa_{\alpha_4}=\frac{3\sqrt{2}}{4}\frac{\sqrt{\left(x_{11}\partial'-x_{12}(\beta_1-\partial')\right)^2+\left(x_{10}\partial'+x_{12}\beta_1\right)^2+\left(x_{11}\beta_1+x_{10}(\beta_1-\partial')\right)^2}}{\mu_1^2\left(\beta_1^2-\beta_1\partial'+\partial'^2\right)\sqrt{\beta_1^2-\beta_1\partial'+\partial'^2}},$$

$$\tau_{\alpha_4}=\frac{\sqrt{3}\left(\begin{array}{l}-(x_{11}\partial'-x_{12}(\beta_1-\partial'))(x_{10}-x_{11}\mu_1\beta_1)\\ +(x_{10}\partial'+x_{12}\beta_1)(x_{11}+x_{10}\mu_1\beta_1-x_{12}\mu_1\partial')\\ -(x_{11}\beta_1+x_{10}(\beta_1-\partial'))(x_{12}+x_{11}\mu_1\partial')\end{array}\right)}{\mu_1\left(\left(x_{11}\partial'-x_{12}(\beta_1-\partial')\right)^2+\left(x_{10}\partial'+x_{12}\beta_1\right)^2+\left(x_{11}\beta_1+x_{10}(\beta_1-\partial')\right)^2\right)}.$$

Proof. By using (28) and (29) and substituting them into (2), we complete the proof. □

Remark 5. *The alternative frame for the $N_1C_1W_1$ Smarandache curve α_4 is established by the following equations:*

$$N_{\alpha_4}=N_{\alpha_4},\qquad C_{\alpha_4}=-\cos\partial_4 T_{\alpha_4}+\sin\partial_4 B_{\alpha_4},\qquad W_{\alpha_4}=\sin\partial_4 T_{\alpha_4}+\cos\partial_4 B_{\alpha_4}$$

where $\partial_4=\measuredangle(B_{\alpha_4},W_{\alpha_4})$, $\cos\partial_4=\dfrac{\kappa_{\alpha_4}}{\sqrt{\kappa_{\alpha_4}^2+\tau_{\alpha_4}^2}}$, *and* $\sin\partial_4=\dfrac{\tau_{\alpha_4}}{\sqrt{\kappa_{\alpha_4}^2+\tau_{\alpha_4}^2}}$.

Example 1. *Let us consider the polynomial general helix curve parameterized as $\zeta(t)=(6s,3s^2,s^3)$. The alternative frame vectors and their corresponding pedal curves according to the point $P=(1,1,1)$ are given as follows:*

$$N=\frac{(-2s,-s^2+2,2s)}{\sqrt{s^2+2}},\quad C=\frac{(s^2-2,-4s,-s^2+2)}{\sqrt{2}(s^2+2)},\qquad W=\frac{(1,0,1)}{\sqrt{2}},$$

$$N-Pedal\Rightarrow\ \alpha_N=\frac{\left(s^4+3s^2+6,2s^2\left(s^4+4s^2+12\right),s^3\left(s^4+6s^2+16\right)\right)}{\left(s^2+2\right)^2},$$

$$C-Pedal\Rightarrow\ \alpha_C=\frac{\left(s\left(s^6+14s^4+52s^2+24\right),2s^2\left(s^4+4s^2-12\right),s\left(s^6+6s^4+4s^2+24\right)\right)}{2\left(s^2+2\right)^2},$$

$$W-Pedal\Rightarrow\ \alpha_W=\frac{\left(4s\left(s^4+6s^2+6\right),s^2\left(3s^4+16s^2+12\right),s^5\left(s^2+4\right)\right)}{\left(s^2+2\right)^2}.$$

In Figure 3, the four of the Smarandache curves of the alternative N-pedal curve according to the point $P(1,1,1)$ are illustrated.

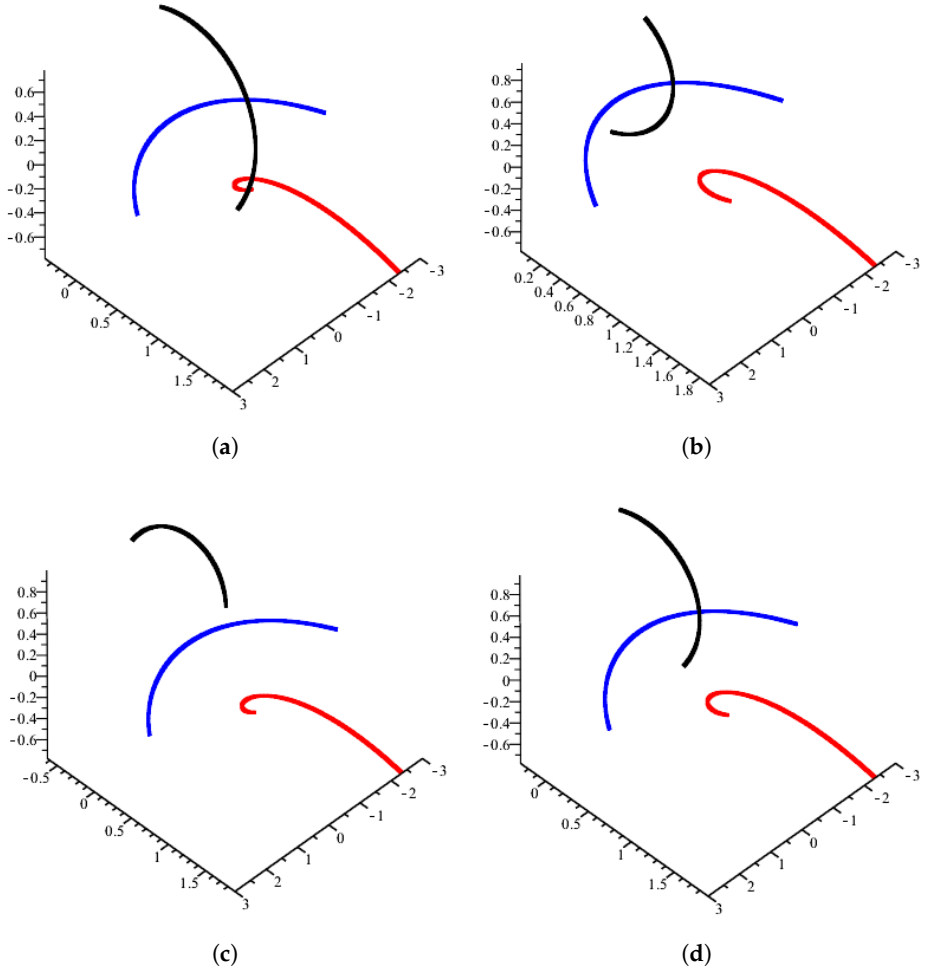

Figure 3. Smarandache curves (black) of the alternative N-pedal curve (red) of the curve $\zeta(t)$ (blue) according to the fixed point $P(1,1,1)$ where $t \in [-0.5, 0.5]$. (**a**) N_1C_1 Smarandache curve; (**b**) N_1W_1-Smarandache curve; (**c**) C_1W_1 Smarandache curve; (**d**) $N_1C_1W_1$ Smarandache curve.

3. The Alternative C-Pedal Curve and Its Smarandache Curves

Similar to the previous section, first, the alternative C-pedal curve resulting from the orthogonal projections of the alternative frame vector C of a given curve according to a fixed point P is defined. Second, we provide the Frenet apparatus of the alternative C-pedal curve and identify the relations among the alternative frame vectors of both the main curve and its corresponding alternative C-pedal curve. Then, we examine four new possible Smarandache curves of the alternative C-pedal curve where the position vectors are taken as the linear combinations of the alternative frame elements. We provide Example 1 and the graphs of these curves.

Definition 6. *Let $\alpha : I \subset R \to R^3$ be a regular curve in E^3 and $\{N, C, W\}$ denote the set of its alternative frame vectors. Then, the locus of the perpendicular projection points of a fixed point P that is not on α onto the alternative frame vector C is called the alternative C-pedal curve for α according to P (Figure 4).*

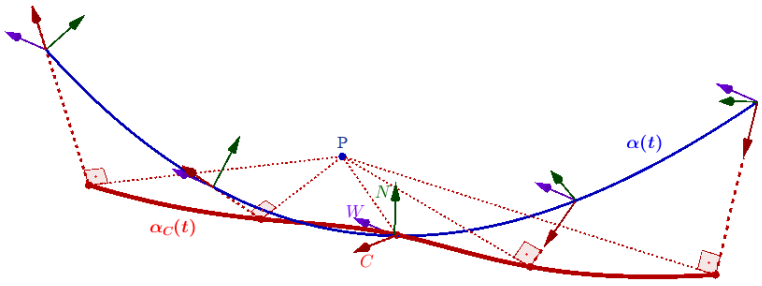

Figure 4. The alternative C-pedal curve (red) of the curve $\alpha(t)$ (blue) according to a fixed point P.

Theorem 12. *The equation of the alternative C-pedal curve of the curve α is given as follows:*

$$\alpha_C(t) = \alpha(t) + \langle P - \alpha(t), C(t)\rangle C(t). \tag{30}$$

Proof. Let P' be the perpendicular projection point onto the vector $\overrightarrow{C}$ from the fixed point P that is not on the curve α. By using the definition of vector projections, we compute the perpendicular projection vector denoted by $\overrightarrow{\alpha P'}$ as follows:

$$\overrightarrow{\alpha P'} = \frac{\left\langle \overrightarrow{\alpha P}, \overrightarrow{C} \right\rangle}{\left\langle \overrightarrow{C}, \overrightarrow{C} \right\rangle} \overrightarrow{C}.$$

Thus, with the motion of the frame vectors along the curve, the geometric location of point P' constructs a new curve that is called the alternative C-pedal curve and denoted by $\alpha_C(t)$. Accordingly, we have the following equations.

$$\begin{aligned} \overrightarrow{\alpha P} = \overrightarrow{\alpha P'} + \overrightarrow{P'P} \Rightarrow \overrightarrow{\alpha P} &= \frac{\left\langle \overrightarrow{\alpha P}, \overrightarrow{C} \right\rangle}{\left\| \overrightarrow{C} \right\|^2} \overrightarrow{C} + \overrightarrow{P'P} \\ &\Rightarrow P' = \alpha(t) + \left\langle \overrightarrow{\alpha P}, \overrightarrow{C} \right\rangle \overrightarrow{C} \\ &\Rightarrow \alpha_C(t) = \alpha(t) + \langle P - \alpha(t), C(t)\rangle C(t), \end{aligned}$$

by which the proof is completed. □

If $\langle P - \alpha(t), C(t)\rangle = \chi(t)$, then Relation (30) can be expressed as follows:

$$\alpha_C(t) = \alpha(t) + \chi(t)C(t). \tag{31}$$

Theorem 13. *Let α_C be the alternative C-pedal curve of a unit speed curve α, and $\{T_2,\ N_2,\ B_2\}$ denote the Frenet vectors of the alternative C-pedal curve. Then, we have the following relations:*

$$\begin{aligned} T_2 =& -\omega_2\chi\beta N + \omega_2(\chi' - v\cos\varphi)C + \omega_2(v\chi\varphi' + v\sin\varphi)W \\ N_2 =& B_2 \wedge T_2 \\ B_2 =& \eta_2(y_3(\chi' - v\cos\varphi) - y_2(v\chi\varphi' + v\sin\varphi))N + \eta_2(y_3\chi\beta + y_1(v\chi\varphi' + v\sin\varphi))C \\ & - \eta_2(y_2\chi\beta + y_1(\chi' - v\cos\varphi))W \end{aligned}$$

where

$$\begin{aligned}
y_1 =& -\beta\Big(\chi + v\chi' - v^2\cos\varphi\Big), \qquad y_2 = \chi' - v\cos\varphi - \chi v\beta^2 - v^2\varphi'(\chi\varphi' + \sin\varphi),\\
y_3 =& v\varphi'(\chi' - v\cos\varphi) + v(\chi\varphi' + \sin\varphi),\\
\omega_2 =& \frac{1}{\sqrt{(\chi\beta)^2 + (\chi' - v\cos\varphi)^2 + (v\chi\varphi' + v\sin\varphi)^2}},\\
\eta_2 =& \frac{1}{\sqrt{\begin{array}{l}(y_3(\chi' - v\cos\varphi) - y_2(v\chi\varphi' + v\sin\varphi))^2 + (y_3\chi\beta + y_1(v\chi\varphi' + v\sin\varphi))^2\\ +(y_2\chi\beta + y_1(\chi' - v\cos\varphi))^2\end{array}}}.
\end{aligned}$$

Proof. By considering the relations in (5), we have the derivatives up to the third degree for Relation (31) as follows:

$$\begin{aligned}
\alpha_C' =& -\chi\beta N + (\chi' - v\cos\varphi)C + (v\chi\varphi' + v\sin\varphi)W,\\
\alpha_C'' =& y_1 N + y_2 C + y_3 W,\\
\alpha_C''' =& (y_1' - vy_2\beta)N + (y_2' + y_1 v\beta - y_3 v\varphi')C + (y_3' + y_2 v\varphi')W.
\end{aligned} \tag{32}$$

Further, if we perform the required vector product operations and take the norms, the following relations are obtained:

$$\begin{aligned}
\alpha_C' \wedge \alpha_C'' =& (y_3(\chi' - v\cos\varphi) - y_2(v\chi\varphi' + v\sin\varphi))N + (y_3\chi\beta + y_1(v\chi\varphi' + v\sin\varphi))C\\
&- (y_2\chi\beta + y_1(\chi' - v\cos\varphi))W,\\
\det(\alpha_C', \alpha_C'', \alpha_C''') =& (y_3(\chi' - v\cos\varphi) - y_2(v\chi\varphi' + v\sin\varphi))(y_1' - vy_2\beta)\\
&+ (y_3\chi\beta + y_1(v\chi\varphi' + v\sin\varphi))(y_2' + y_1 v\beta - y_3 v\varphi')\\
&- (y_2\chi\beta + y_1(\chi' - v\cos\varphi))(y_3' + y_2 v\varphi')
\end{aligned} \tag{33}$$

$$\begin{aligned}
\|\alpha_C'\| =& \sqrt{(\chi\beta)^2 + (\chi' - v\cos\varphi)^2 + (v\chi\varphi' + v\sin\varphi)^2},\\
\|\alpha_C' \wedge \alpha_C''\| =& \sqrt{\begin{array}{l}(y_3(\chi' - v\cos\varphi) - y_2(v\chi\varphi' + v\sin\varphi))^2 + (y_3\chi\beta + y_1(v\chi\varphi' + v\sin\varphi))^2\\ +(y_2\chi\beta + y_1(\chi' - v\cos\varphi))^2\end{array}}.
\end{aligned} \tag{34}$$

If Equations (32)–(34) are substituted in (1), the proof is completed. □

Theorem 14. *Let α_C be the alternative C-pedal curve of the unit speed curve α, and κ_2 and τ_2 denote its curvature and torsion functions, respectively. Then, we have the following relations among the curvatures:*

$$\begin{aligned}
\kappa_2 =& \frac{\sqrt{\begin{array}{l}(y_3(\chi' - v\cos\varphi) - y_2(v\chi\varphi' + v\sin\varphi))^2 + (y_3\chi\beta + y_1(v\chi\varphi' + v\sin\varphi))^2\\ +(y_2\chi\beta + y_1(\chi' - v\cos\varphi))^2\end{array}}}{\Big((\chi\beta)^2 + (\chi' - v\cos\varphi)^2 + (v\chi\varphi' + v\sin\varphi)^2\Big)^{\frac{3}{2}}},\\
\tau_2 =& \frac{\begin{array}{l}(y_3(\chi' - v\cos\varphi) - y_2(v\chi\varphi' + v\sin\varphi))(y_1' - vy_2\beta)\\ +(y_3\chi\beta + y_1(v\chi\varphi' + v\sin\varphi))(y_2' + y_1 v\beta - y_3 v\varphi')\\ -(y_2\chi\beta + y_1(\chi' - v\cos\varphi))(y_3' + y_2 v\varphi')\end{array}}{\begin{array}{l}(y_3(\chi' - v\cos\varphi) - y_2(v\chi\varphi' + v\sin\varphi))^2 + (y_3\chi\beta + y_1(v\chi\varphi' + v\sin\varphi))^2\\ +(y_2\chi\beta + y_1(\chi' - v\cos\varphi))^2\end{array}}.
\end{aligned}$$

Proof. We substitute the relations given in (33) and (34) in (2) and complete the proof. □

Remark 6. *The alternative frame for the alternative C-pedal curve α_C is established by the following equations:*

$$N_2 = N_2, \qquad C_2 = -\cos\Delta T_2 + \sin\Delta B_2, \qquad W_2 = \sin\Delta T_2 + \cos\Delta B_2, \tag{35}$$

where $\Delta = \measuredangle(B_2, W_2)$, $\cos\Delta = \dfrac{\kappa_2}{\sqrt{\kappa_2^2+\tau_2^2}}$, *and* $\sin\Delta = \dfrac{\tau_2}{\sqrt{\kappa_2^2+\tau_2^2}}$.

Corollary 2. *The alternative frame formulae for the alternative C-pedal curve is given as follows:*

$$N_2' = \mu_2\beta_2 C_2, \qquad C'_2 = -\mu_2\beta_2 N_2 + \mu_2\Delta' W_2, \qquad W_2{}' = -\mu_2\Delta' C_2, \tag{36}$$

where $\mu_2 = \|\alpha_C'\|$, $\Delta' = \left(\dfrac{\tau_2}{\kappa_2}\right)'\left(\dfrac{\kappa_2^2+\tau_2^2}{\kappa_2^2}\right)$, $\beta_2 = \mu_2\sqrt{\kappa_2^2+\tau_2^2}$.

Definition 7. *The N_2C_2 Smarandache curve of the alternative C-pedal curve whose position vector is the linear combination of the two vectors N_2 and C_2 is defined by the following parametrization:*

$$\delta_1 = \frac{N_2 + C_2}{\sqrt{2}}. \tag{37}$$

Theorem 15. *Let $\{N_{\delta_1}, C_{\delta_1}, W_{\delta_1}\}$ denote the alternative frame for the N_2C_2 Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$\begin{aligned}
T_{\delta_1} =& \frac{\sqrt{2}(-\beta_2 N_2 + \beta_2 C_2 + \Delta' W_2)}{\sqrt{2\beta_2^2 + \Delta'^2}},\\
N_{\delta_1} =& B_{\delta_1} \wedge T_{\delta_1},\\
B_{\delta_1} =& \frac{(y_6\beta_1 - y_5\Delta')N_2 + (y_6\beta_2 + y_4\Delta')C_2 - (\beta_2 y_4 + \beta_2 y_5)W_2}{\sqrt{(y_6\beta_1 - y_5\Delta')^2 + (y_6\beta_2 + y_4\Delta')^2 + (\beta_2 y_4 + \beta_2 y_5)^2}},
\end{aligned}$$

where $y_4 = -\dfrac{(\mu_2\beta_2)' + \mu_2^2\beta_2^2}{\sqrt{2}}$, $y_5 = \dfrac{(\mu_2\beta_2)' - \mu_2^2\left(\beta_2^2 + \Delta'^2\right)}{\sqrt{2}}$, $y_6 = \dfrac{(\mu_2\Delta')' + \mu_2^2\beta_2\Delta'}{\sqrt{2}}$.

Proof. If Relation (36) is taken into account, and the required derivatives for the N_2C_2 curve given in (37) are taken, we have

$$\begin{aligned}
\delta_1' =& \frac{-\mu_2\beta_2 N_2 + \mu_2\beta_2 C_2 + \mu_2\Delta' W_2}{\sqrt{2}},\\
\delta_1'' =& y_4 N_2 + y_5 C_2 + y_6 W_2,\\
\delta_1''' =& (y_4' - y_5\mu_2\beta_2)N_2 + (y_5' + y_4\mu_2\beta_2 - y_6\mu_2\Delta')C_2 + (y_6' + y_5\mu_2\Delta')W_2,
\end{aligned} \tag{38}$$

Moreover, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}
\delta_1' \wedge \delta_1'' =& \frac{\mu_2}{\sqrt{2}}\left((y_6\beta_2 - y_5\Delta')N_2 + (y_6\beta_2 + y_4\Delta')C_2 - \beta_2(y_4 + y_5)W_2\right),\\
\det(\delta_1', \delta_1'', \delta'''_1) =& \frac{\mu_2}{\sqrt{2}}\begin{pmatrix} (y_4' - y_5\mu_2\beta_2)(y_6\beta_2 - y_5\Delta') \\ +(y_5' + y_4\mu_2\beta_2 - y_6\mu_2\Delta')(y_6\beta_2 + y_4\Delta') \\ -\beta_2(y_4 + y_5)(y_6' + y_5\mu_2\Delta') \end{pmatrix},
\end{aligned} \tag{39}$$

and

$$\begin{aligned}\|\delta_1'\| =& \frac{\mu_2}{\sqrt{2}}\sqrt{2\beta_2^2+\Delta'^2},\\ \|\delta_1'\wedge\delta_1''\| =& \frac{\mu_2}{\sqrt{2}}\sqrt{(y_6\beta_2-y_5\Delta')^2+(y_6\beta_1+y_4\Delta')^2+(\beta_2y_4+\beta_2y_5)^2}\ .\end{aligned} \tag{40}$$

Finally, by substituting Relations (38)–(40) into (1), we complete the proof. □

Theorem 16. *Let κ_{δ_1} and τ_{δ_1} be the curvature and the torsion functions for the N_2C_2 Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\begin{aligned}\kappa_{\delta_1} =& \frac{2\sqrt{(y_6\beta_2-y_5\Delta')^2+(y_6\beta_1+y_4\Delta')^2+(\beta_2y_4+\beta_2y_5)^2}}{\mu_2^2\left(2\beta_2^2+\Delta'^2\right)\sqrt{2\beta_2^2+\Delta'^2}},\\ \tau_{\delta_1} =& \frac{\sqrt{2}\left(\begin{array}{c}(y_4'-y_5\mu_2\beta_2)(y_6\beta_2-y_5\Delta')+(y_5'+y_4\mu_2\beta_2-y_6\mu_2\Delta')(y_6\beta_2+y_4\Delta')\\ -\beta_2(y_4+y_5)(y_6'+y_5\mu_2\Delta')\end{array}\right)}{\mu_2\left((y_6\beta_2-y_5\Delta')^2+(y_6\beta_1+y_4\Delta')^2+(\beta_2y_4+\beta_2y_5)^2\right)}.\end{aligned}$$

Proof. By using (39) and (40) and substituting them into (2), we complete the proof. □

Remark 7. *The alternative frame for the N_2C_2 Smarandache curve δ_1 is established by the following equations:*

$$N_{\delta_1}=N_{\delta_1},\quad C_{\delta_1}=-\cos\Delta_1T_{\delta_1}+\sin\Delta_1B_{\delta_1},\quad W_{\delta_1}=\sin\Delta_1T_{\delta_1}+\cos\Delta_1B_{\delta_1},$$

where $\Delta_1=\measuredangle\left(B_{\delta_1},W_{\delta_1}\right)$, $\cos\Delta_1=\dfrac{\kappa_{\delta_1}}{\sqrt{\kappa_{\delta_1}^2+\tau_{\delta_1}^2}}$, *and* $\sin\Delta_1=\dfrac{\tau_{\delta_1}}{\sqrt{\kappa_{\delta_1}^2+\tau_{\delta_1}^2}}$.

Definition 8. *The N_2W_2 Smarandache curve of the alternative C-pedal curve whose position vector is the linear combination of the two vectors N_2 and W_2 is defined by the following parametrization:*

$$\delta_2=\frac{N_2+W_2}{\sqrt{2}}. \tag{41}$$

Theorem 17. *Let $\{N_{\delta_2},\ C_{\delta_2},\ W_{\delta_2}\}$ denote the alternative frame for the N_2W_2 Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$T_{\delta_2}=C_2,\quad N_{\delta_2}=\frac{-\beta_2N_2+\Delta'W_2}{\sqrt{\Delta'^2+\beta_2^2}},\quad B_{\delta_2}=\frac{\Delta'N_2+\beta_2W_2}{\sqrt{\Delta'^2+\beta_2^2}}.$$

Proof. If Relation (36) is taken into account, and the required derivatives for the N_2W_2 curve given in (41) are taken, we have

$$\begin{aligned}\delta_2' =& \frac{\mu_2(\beta_2-\Delta')C_2}{\sqrt{2}},\\ \delta_2'' =& \frac{-\mu_2^2\beta_2(\beta_2-\Delta')N_2+(\mu_2(\beta_2-\Delta'))'C_2+\mu_2^2\Delta'(\beta_2-\Delta')W_2}{\sqrt{2}},\\ \delta'''_2 =& \frac{\begin{array}{l}-\left((\mu_2^2\beta_2(\beta_2-\Delta'))'+\mu_2\beta_2((\mu_2\beta_2-\mu_2\Delta'))'\right)N_2\\ +\left(((\mu_2\beta_2-\mu_2\Delta'))''-\mu_2^3\left(\beta_2^2+\Delta'^2\right)(\beta_2-\Delta')\right)C_2\\ +\left(\mu_2\Delta'(\mu_2(\beta_2-\Delta'))'+(\mu_2^2\Delta'(\beta_2-\Delta'))'\right)W_2\end{array}}{\sqrt{2}}.\end{aligned} \tag{42}$$

Moreover, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}\delta_2' \wedge \delta_2'' =& \frac{\mu_2^3 \Delta'(\beta_2 - \Delta')^2 N_2 + \mu_2^3 \beta_2 (\beta_2 - \Delta')^2 W_2}{2},\\ \det(\delta_2', \delta_2'', \delta_2''') =& \frac{\mu_2^5 (\beta_2 - \Delta')^3 (\beta_2 \Delta'' - \Delta' \beta_2')}{2\sqrt{2}},\end{aligned} \tag{43}$$

and

$$\begin{aligned}\left\|\delta_2'\right\| =& \frac{\mu_2(\beta_2 - \Delta')}{\sqrt{2}},\\ \left\|\delta_2' \wedge \delta_2''\right\| =& \frac{(\beta_2 - \Delta')^2 \mu_2^3 \sqrt{\Delta'^2 + \beta_2^2}}{2}.\end{aligned} \tag{44}$$

Finally, by substituting Relations (42)–(44) into (1), we complete the proof. □

Theorem 18. *Let κ_{δ_2} and τ_{δ_2} be the curvature and the torsion functions for the N_2W_2 Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\kappa_{\delta_2} = \frac{\sqrt{2}\sqrt{\Delta'^2 + \beta_2^2}}{\beta_2 - \Delta'}, \qquad \tau_{\delta_2} = \frac{\sqrt{2}(\beta_2 \Delta'' - \Delta' \beta_2')}{\mu_2(\beta_2 - \Delta')\left(\Delta'^2 + \beta_2^2\right)}.$$

Proof. By using (43) and (44) and substituting them into (2), we complete the proof. □

Remark 8. *The alternative frame for the N_2W_2 Smarandache curve δ_2 is established by the following equations:*

$$N_{\delta_2} = N_{\delta_2}, \quad C_{\delta_2} = -\cos\Delta_2 T_{\delta_2} + \sin\Delta_2 B_{\delta_2}, \quad W_{\delta_2} = \sin\Delta_2 T_{\delta_2} + \cos\Delta_2 B_{\delta_2},$$

where $\Delta_2 = \measuredangle(B_{\delta_2}, W_{\delta_2})$, $\cos\Delta_2 = \dfrac{\kappa_{\delta_2}}{\sqrt{\kappa_{\delta_2}^2 + \tau_{\delta_2}^2}}$, *and* $\sin\Delta_2 = \dfrac{\tau_{\delta_2}}{\sqrt{\kappa_{\delta_2}^2 + \tau_{\delta_2}^2}}$.

Definition 9. *The C_2W_2 Smarandache curve of the alternative C-pedal curve whose position vector is the linear combination of the two vectors C_2 and W_2 is defined by the following parametrization:*

$$\delta_3 = \frac{C_2 + W_2}{\sqrt{2}}. \tag{45}$$

Theorem 19. *Let $\{N_{\delta_3}, C_{\delta_3}, W_{\delta_3}\}$ denote the alternative frame for the C_2W_2 Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$\begin{aligned}T_{\delta_3} =& \frac{-\beta_2 N_2 - \Delta' C_2 + \Delta' W_2}{\sqrt{\beta_2^2 + 2\Delta'^2}},\\ N_{\delta_3} =& B_{\delta_3} \wedge T_{\delta_3},\\ B_{\delta_3} =& \frac{-(\Delta' y_9 + \Delta' y_8) N_2 + (\beta_2 y_9 + \Delta' y_7) C_2 + (\Delta' y_7 - \beta_2 y_8) W_2}{\sqrt{(\Delta' y_9 + \Delta' y_8)^2 + (\beta_2 y_9 + \Delta' y_7)^2 + (\Delta' y_7 - \beta_2 y_8)^2}},\end{aligned}$$

where $y_7 = \dfrac{\mu_2^2 \Delta' \beta_2 - (\mu_2 \beta_2)'}{\sqrt{2}}$, $y_8 = -\dfrac{(\mu_2 \Delta')' + \mu_2^2\left(\beta_2^2 + \Delta'^2\right)}{\sqrt{2}}$, $y_9 = \dfrac{(\mu_2 \Delta')' - \mu_2^2 \Delta'^2}{\sqrt{2}}$.

Proof. If Relation (36) is taken into account, and the required derivatives for the C_2W_2 curve given in (45) are taken, we have

$$\begin{aligned}\delta_3' =& \frac{-\mu_2\beta_2 N_2 - \mu_2\Delta' C_2 + \mu_2\Delta' W_2}{\sqrt{2}},\\ \delta_3'' =& y_7 N_2 + y_8 C_2 + y_9 W_2,\\ \delta_3''' =& \left(y_7' - y_8\mu_2\beta_2\right)N_2 + \left(y_8' + y_7\mu_2\beta_2 - y_9\mu_2\Delta'\right)C_2 + \left(y_9' + y_8\mu_2\Delta'\right)W_2.\end{aligned} \tag{46}$$

Moreover, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}\delta_3' \wedge \delta_3'' =& \frac{\mu_2(-\Delta'(y_9+y_8)N_2 + (\beta_2 y_9 + \Delta' y_7)C_2 + (\Delta' y_7 - \beta_2 y_8)W_2)}{\sqrt{2}},\\ \det(\delta_3', \delta_3'', \delta_3''') =& \frac{\mu_2}{\sqrt{2}}\begin{pmatrix} (\Delta' y_7 - \beta_2 y_8)(y_9' + y_8\mu_2\Delta') - \Delta'(y_9+y_8)(y_7' - y_8\mu_2\beta_2)\\ +(\beta_2 y_9 + \Delta' y_7)(y_8' + y_7\mu_2\beta_2 - y_9\mu_2\Delta')\end{pmatrix},\end{aligned} \tag{47}$$

and

$$\begin{aligned}\left\|\delta_3'\right\| =& \frac{\mu_2}{\sqrt{2}}\sqrt{\beta_2^2 + 2\Delta'^2},\\ \left\|\delta_3' \wedge \delta_3''\right\| =& \frac{\mu_2}{\sqrt{2}}\sqrt{\left(\Delta' y_9 + \Delta' y_8\right)^2 + \left(\beta_2 y_9 + \Delta' y_7\right)^2 + \left(\Delta' y_7 - \beta_2 y_8\right)^2}\ .\end{aligned} \tag{48}$$

Finally, by substituting Relations (46)–(48) into (1), we complete the proof. □

Theorem 20. *Let κ_{δ_3} and τ_{δ_3} be the curvature and the torsion functions for the C_2W_2 Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\begin{aligned}\kappa_{\delta_3} =& \frac{2\sqrt{\left(\Delta' y_9 + \Delta' y_8\right)^2 + \left(\beta_2 y_9 + \Delta' y_7\right)^2 + \left(\Delta' y_7 - \beta_2 y_8\right)^2}}{\mu_2^2\left(\beta_2^2 + 2\Delta'^2\right)\sqrt{\beta_2^2 + 2\Delta'^2}},\\ \tau_{\delta_3} =& \frac{\sqrt{2}\begin{pmatrix} (\Delta' y_7 - \beta_2 y_8)(y_9' + y_8\mu_2\Delta') - \Delta'(y_9+y_8)(y_7' - y_8\mu_2\beta_2)\\ +(\beta_2 y_9 + \Delta' y_7)(y_8' + y_7\mu_2\beta_2 - y_9\mu_2\Delta')\end{pmatrix}}{\mu_2\left(\left(\Delta' y_9 + \Delta' y_8\right)^2 + \left(\beta_2 y_9 + \Delta' y_7\right)^2 + \left(\Delta' y_7 - \beta_2 y_8\right)^2\right)}.\end{aligned}$$

Proof. By using (47) and (48) and substituting them into (2), we complete the proof. □

Remark 9. *The alternative frame for the C_2W_2 Smarandache curve δ_3 is established by the following equations:*

$$N_{\delta_3} = N_{\delta_3}, \quad C_{\delta_3} = -\cos\Delta_3 T_{\delta_3} + \sin\Delta_3 B_{\delta_3}, \quad W_{\delta_3} = \sin\Delta_3 T_{\delta_3} + \cos\Delta_3 B_{\delta_3},$$

where $\Delta_3 = \measuredangle\left(B_{\delta_3}, W_{\delta_3}\right)$, $\cos\Delta_3 = \dfrac{\kappa_{\delta_3}}{\sqrt{\kappa_{\delta_3}^2 + \tau_{\delta_3}^2}}$, *and* $\sin\Delta_3 = \dfrac{\tau_{\delta_3}}{\sqrt{\kappa_{\delta_3}^2 + \tau_{\delta_3}^2}}$.

Definition 10. *The $N_2C_2W_2$ Smarandache curve of the alternative C-pedal curve whose position vector is the linear combination of the vectors N_2, C_2 and W_2 is defined by the following parametrization:*

$$\delta_4 = \frac{N_2 + C_2 + W_2}{\sqrt{3}}. \tag{49}$$

Theorem 21. *Let* $\{N_{\delta_4},\ C_{\delta_4},\ W_{\delta_4}\}$ *denote the alternative frame for the* $N_2C_2W_2$ *Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$\begin{aligned}
T_{\delta_4} &= \tfrac{-\beta_2 N_2+(\beta_2-\Delta')C_2+\Delta' W_2}{\sqrt{2}\sqrt{\beta_2^2-\beta_2\Delta'+\Delta'^2}},\\
N_{\delta_4} &= B_{\alpha_4}\wedge T_{\delta_4},\\
B_{\delta_4} &= \tfrac{-(y_{11}\Delta'-y_{12}(\beta_2-\Delta'))N_2+(y_{10}\Delta'+y_{12}\beta_2)C_2-(y_{11}\beta_2+y_{10}(\beta_2-\Delta'))W_2}{\sqrt{(y_{11}\Delta'-y_{12}(\beta_2-\Delta'))^2+(y_{10}\Delta'+y_{12}\beta_2)^2+(y_{11}\beta_2+y_{10}(\beta_2-\Delta'))^2}}
\end{aligned}$$

$$\textit{where}\ \ y_{10} = \frac{(\mu_1\beta_1)'+\mu_1^2\beta_1(\beta_1-\partial')}{\sqrt{3}},\qquad y_{11} = \frac{(\mu_1(\beta_1-\partial'))'-\mu_1^2\left(\beta_1^2+(\partial_1')^2\right)}{\sqrt{3}},$$
$$y_{12} = \frac{(\mu_1\partial')'+\mu_1^2\partial'(\beta_1-\partial')}{\sqrt{3}}.$$

Proof. If Relation (36) is taken into account, and the required derivatives for the $N_2C_2W_2$ curve given in (49) are taken, we have

$$\begin{aligned}
\delta_4' &= \frac{\mu_2(-\beta_2N_2+(\beta_2-\Delta')C_2+\Delta'W_2)}{\sqrt{3}},\\
\delta_4'' &= y_{10}N_2+y_{11}C_2+y_{12}W_2,\\
\delta_4''' &= (y_{10}-y_{11}\mu_2\beta_2)N_2+\left(y_{11}+y_{10}\mu_2\beta_2-y_{12}\mu_2\Delta'\right)C_2+\left(y_{12}+y_{11}\mu_2\Delta'\right)W_2.
\end{aligned}\tag{50}$$

Moreover, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}
\delta_4'\wedge\delta_4'' &= \frac{\mu_2\begin{pmatrix}-(y_{11}\Delta'-y_{12}(\beta_2-\Delta'))N_2+(y_{10}\Delta'+y_{12}\beta_2)C_2\\ -(y_{11}\beta_2+y_{10}(\beta_2-\Delta'))W_2\end{pmatrix}}{\sqrt{3}},\\
\det(\delta_4',\delta_4'',\delta_4''') &= \frac{\mu_2}{\sqrt{3}}\begin{pmatrix}-(y_{11}\Delta'-y_{12}(\beta_2-\Delta'))(y_{10}-y_{11}\mu_2\beta_2)\\ +(y_{10}\Delta'+y_{12}\beta_2)(y_{11}+y_{10}\mu_2\beta_2-y_{12}\mu_2\Delta')\\ -(y_{11}\beta_2+y_{10}(\beta_2-\Delta'))(y_{12}+y_{11}\mu_2\Delta')\end{pmatrix},
\end{aligned}\tag{51}$$

and

$$\begin{aligned}
\|\delta_4'\| &= \frac{\sqrt{6}\mu_2}{3}\sqrt{\beta_2^2-\beta_2\Delta'+\Delta'^2},\\
\|\delta_4'\wedge\delta_4''\| &= \frac{\mu_2}{\sqrt{3}}\sqrt{(y_{11}\Delta'-y_{12}(\beta_2-\Delta'))^2+(y_{10}\Delta'+y_{12}\beta_2)^2+(y_{11}\beta_2+y_{10}(\beta_2-\Delta'))^2}.
\end{aligned}\tag{52}$$

Finally, by substituting Relations (50)–(52) into (1), we complete the proof. □

Theorem 22. *Let* κ_{δ_4} *and* τ_{δ_4} *be the curvature and the torsion functions for the* $N_2C_2W_2$ *Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\begin{aligned}
\kappa_{\delta_4} &= \frac{3\sqrt{2}}{4}\frac{\sqrt{(y_{11}\Delta'-y_{12}(\beta_2-\Delta'))^2+(y_{10}\Delta'+y_{12}\beta_2)^2+(y_{11}\beta_2+y_{10}(\beta_2-\Delta'))^2}}{\mu_2^2\left(\beta_2^2-\beta_2\Delta'+\Delta'^2\right)\sqrt{\beta_2^2-\beta_2\Delta'+\Delta'^2}},\\
\tau_{\delta_4} &= \frac{\sqrt{3}\begin{pmatrix}-(y_{11}\Delta'-y_{12}(\beta_2-\Delta'))(y_{10}-y_{11}\mu_2\beta_2)\\ +(y_{10}\Delta'+y_{12}\beta_2)(y_{11}+y_{10}\mu_2\beta_2-y_{12}\mu_2\Delta')\\ -(y_{11}\beta_2+y_{10}(\beta_2-\Delta'))(y_{12}+y_{11}\mu_2\Delta')\end{pmatrix}}{\mu_2\left((y_{11}\Delta'-y_{12}(\beta_2-\Delta'))^2+(y_{10}\Delta'+y_{12}\beta_2)^2+(y_{11}\beta_2+y_{10}(\beta_2-\Delta'))^2\right)}.
\end{aligned}$$

Proof. By using (51) and (52) and substituting them into (2), we complete the proof. □

Remark 10. *The alternative frame for the $N_2C_2W_2$ Smarandache curve δ_4 is established by the following equations:*

$$N_{\delta_4} = N_{\delta_4}, \quad C_{\delta_4} = -\cos\Delta_4 T_{\delta_4} + \sin\Delta_4 B_{\delta_4}, \quad W_{\delta_4} = \sin\Delta_4 T_{\delta_4} + \cos\Delta_4 B_{\delta_4},$$

where $\Delta_4 = \measuredangle(B_{\delta_4}, W_{\delta_4})$, $\cos\Delta_4 = \frac{\kappa_{\delta_4}}{\sqrt{\kappa_{\delta_4}^2+\tau_{\delta_4}^2}}$, *and* $\sin\Delta_4 = \frac{\tau_{\delta_4}}{\sqrt{\kappa_{\delta_4}^2+\tau_{\delta_4}^2}}$.

By using Example 1, Smarandache curves of the alternative C-pedal curve according to the point $P(1,1,1)$ are illustrated in Figure 5.

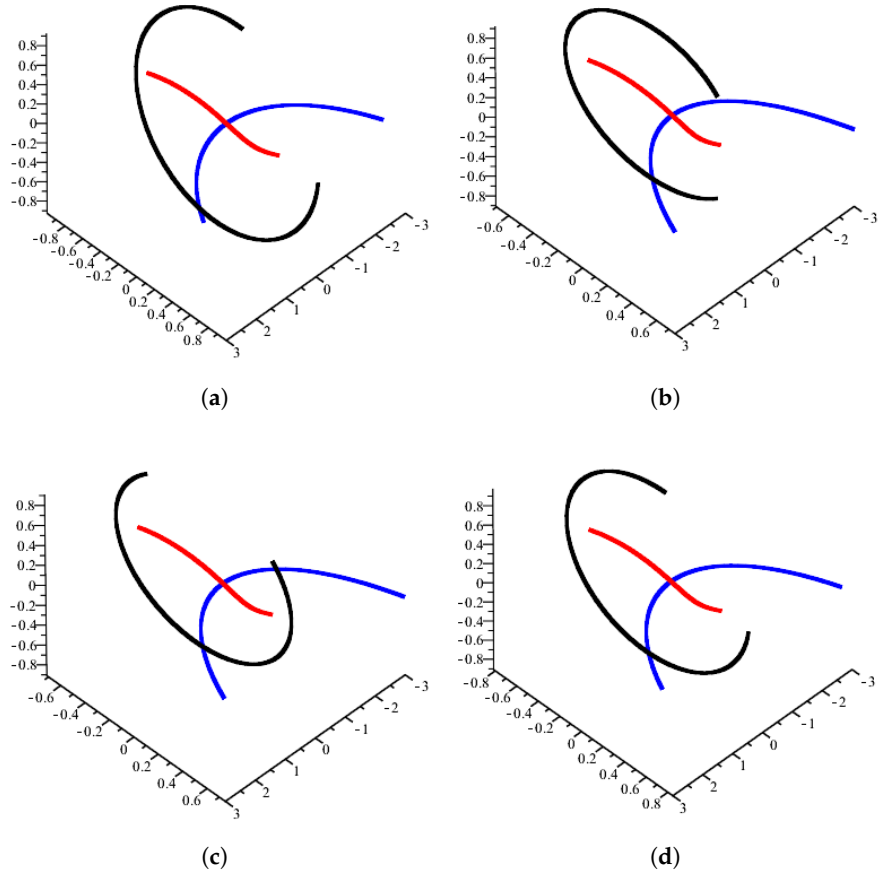

Figure 5. Smarandache curves (black) of the alternative C-pedal curve (red) of the curve $\zeta(t)$ (blue) according to the fixed point $P(1,1,1)$ where $t \in [-0.5, 0.5]$. (**a**) N_2C_2 Smarandache curve; (**b**) N_2W_2 Smarandache curve; (**c**) C_2W_2 Smarandache curve; (**d**) $N_2C_2W_2$ Smarandache curve.

4. The Alternative W-Pedal Curve and Its Smarandache Curves

We deal with the last vector W of alternative frame to define the alternative W-pedal curve in this section and determine the Frenet apparatus of this curve. Similarly, the relations among the alternative frame vectors of both the main curve and the alternative W-pedal curve are computed, and the Smarandache curves of the alternative W-pedal curve are defined. The invariants of each Smarandache curve are expressed by the invariants of the alternative W-pedal curve. Example 1 is applied to the alternative W-pedal curve and each curve is illustrated.

Definition 11. *Let $\alpha : I \subset R \to R^3$ be a regular curve in E^3 and $\{N, C, W\}$ denote the set of its alternative frame vectors. Then, the locus of the perpendicular projection points of a fixed point P that is not on α onto the alternative frame vector W is called the alternative W-pedal curve for α according to P (Figure 6).*

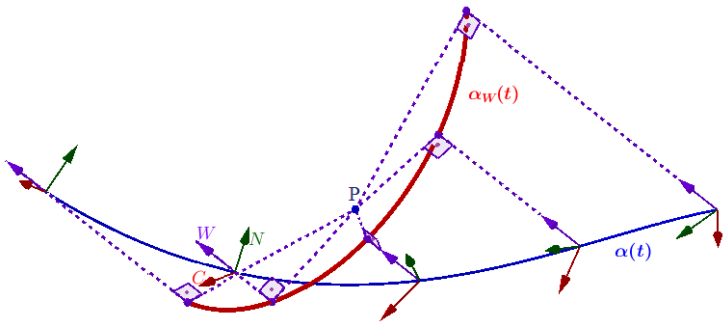

Figure 6. The alternative W-pedal curve (red) of the curve $\alpha(t)$ (blue) according to a fixed point P.

Theorem 23. *The equation of the alternative W-pedal curve of the curve α is given as follows:*

$$\alpha_W(t) = \alpha(t) + \langle P - \alpha(t), W(t)\rangle W(t). \tag{53}$$

Proof. Let P' be the perpendicular projection point onto the vector $\overrightarrow{W}$ from the fixed point P that is not on the curve α. By using the definition of vector projections, we compute the perpendicular projection vector denoted by $\overrightarrow{\alpha P'}$ as follows:

$$\overrightarrow{\alpha P'} = \frac{\left\langle \overrightarrow{\alpha P}, \overrightarrow{W} \right\rangle}{\left\langle \overrightarrow{W}, \overrightarrow{W} \right\rangle} \overrightarrow{W}.$$

Thus, with the motion of the frame vectors along the curve, the geometric location of point P' constructs a new curve that is called the alternative W-pedal curve and denoted by $\alpha_W(t)$. Accordingly, we have the following equations:

$$\begin{aligned} \overrightarrow{\alpha P} = \overrightarrow{\alpha P'} + \overrightarrow{P'P} \Rightarrow \overrightarrow{\alpha P} &= \frac{\left\langle \overrightarrow{\alpha P}, \overrightarrow{W} \right\rangle}{\left\| \overrightarrow{W} \right\|^2} \overrightarrow{W} + \overrightarrow{P'P} \\ &\Rightarrow P' = \alpha(t) + \left\langle \overrightarrow{\alpha P}, \overrightarrow{W} \right\rangle \overrightarrow{W} \\ &\Rightarrow \alpha_W(t) = \alpha(t) + \langle P - \alpha(t), W(t)\rangle W(t), \end{aligned}$$

by which the proof is completed. □

If $\langle P - \alpha(t), W(t)\rangle = \varsigma(t)$, then Relation (53) can be expressed as follows:

$$\alpha_W(t) = \alpha(t) + \varsigma(t) W(t). \tag{54}$$

Theorem 24. *Let α_W be the alternative W-pedal curve of a unit speed curve α, and $\{T_3,\ N_3,\ B_3\}$ denote the Frenet vectors of the alternative W-pedal curve. Then, we have the following relations:*

$$\begin{aligned} T_3 =& -\omega_3 \varsigma \beta N + \omega_3 \left(\varsigma' - v\cos\varphi\right) C + \omega_3 \left(v\varsigma\varphi' + v\sin\varphi\right) W, \\ N_3 =& B_3 \wedge T_3, \\ B_3 =& \eta_3 \left(z_3 \left(\varsigma' - v\cos\varphi\right) - z_2 \left(v\varsigma\varphi' + v\sin\varphi\right)\right) N + \eta_2 \left(z_3\varsigma\beta + z_1 \left(v\varsigma\varphi' + v\sin\varphi\right)\right) C \\ & - \eta_3 \left(z_2\varsigma\beta + z_1 \left(\varsigma' - v\cos\varphi\right)\right) W, \end{aligned}$$

where

$$
\begin{aligned}
z_1 =& -\beta\Big(\varsigma + v\varsigma' - v^2\cos\varphi\Big), \qquad z_2 = \varsigma' - v\cos\varphi - \varsigma v\beta^2 - v^2\varphi'\big(\varsigma\varphi' + \sin\varphi\big),\\
z_3 =& v\varphi'\big(\varsigma' - v\cos\varphi\big) + v\big(\varsigma\varphi' + \sin\varphi\big),\\
\omega_3 =& \frac{1}{\sqrt{(\varsigma\beta)^2 + (\varsigma' - v\cos\varphi)^2 + (v\varsigma\varphi' + v\sin\varphi)^2}},\\
\eta_3 =& \frac{1}{\sqrt{\begin{array}{l}(z_3(\varsigma' - v\cos\varphi) - z_2(v\varsigma\varphi' + v\sin\varphi))^2 + (z_3\varsigma\beta + z_1(v\varsigma\varphi' + v\sin\varphi))^2\\ +(z_2\varsigma\beta + z_1(\varsigma' - v\cos\varphi))^2\end{array}}}.
\end{aligned}
$$

Proof. By considering the relations in (5), we have the derivatives up to the third degree for Relation (54) as follows:

$$
\begin{aligned}
\alpha_W' =& -\varsigma\beta N + \big(\varsigma' - v\cos\varphi\big)C + \big(v\varsigma\varphi' + v\sin\varphi\big)W,\\
\alpha_W'' =& z_1 N + z_2 C + z_3 W,\\
\alpha_W''' =& \big(z_1' - vz_2\beta\big)N + \big(z_2' + z_1 v\beta - z_3 v\varphi'\big)C + \big(z_3' + z_2 v\varphi'\big)W,
\end{aligned}
\tag{55}
$$

Further, if we perform the required vector product operations and take the norms, the following relations are obtained.

$$
\begin{aligned}
\alpha_W' \wedge \alpha_W'' =& \big(z_3\big(\varsigma' - v\cos\varphi\big) - z_2\big(v\varsigma\varphi' + v\sin\varphi\big)\big)N\\
&+ \big(z_3\varsigma\beta + z_1\big(v\varsigma\varphi' + v\sin\varphi\big)\big)C\\
&- \big(z_2\varsigma\beta + z_1\big(\varsigma' - v\cos\varphi\big)\big)W,\\
\det\big(\alpha_W', \alpha_W'', \alpha_W'''\big) =& \big(z_3\big(\varsigma' - v\cos\varphi\big) - z_2\big(v\varsigma\varphi' + v\sin\varphi\big)\big)\big(z_1' - vz_2\beta\big)\\
&+ \big(z_3\varsigma\beta + z_1\big(v\varsigma\varphi' + v\sin\varphi\big)\big)\big(z_2' + z_1 v\beta - z_3 v\varphi'\big)\\
&- \big(z_2\varsigma\beta + z_1\big(\varsigma' - v\cos\varphi\big)\big)\big(z_3' + z_2 v\varphi'\big),
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
\|\alpha_W'\| =& \sqrt{(\varsigma\beta)^2 + (\varsigma' - v\cos\varphi)^2 + (v\varsigma\varphi' + v\sin\varphi)^2},\\
\|\alpha_W' \wedge \alpha_W''\| =& \sqrt{\begin{array}{l}(z_3(\varsigma' - v\cos\varphi) - z_2(v\varsigma\varphi' + v\sin\varphi))^2 + (z_3\varsigma\beta + z_1(v\varsigma\varphi' + v\sin\varphi))^2\\ +(z_2\varsigma\beta + z_1(\varsigma' - v\cos\varphi))^2\end{array}}.
\end{aligned}
\tag{57}
$$

If Equations (55)–(57) are substituted in (1), the proof is completed. □

Theorem 25. *Let α_W be the alternative W-pedal curve of the unit speed curve α, and κ_3 and τ_3 denote its curvature and torsion functions, respectively. Then, we have the following relations among the curvatures.*

$$
\begin{aligned}
\kappa_3 =& \frac{\sqrt{\begin{array}{l}(z_3(\varsigma' - v\cos\varphi) - z_2(v\varsigma\varphi' + v\sin\varphi))^2 + (z_3\varsigma\beta + z_1(v\varsigma\varphi' + v\sin\varphi))^2\\ +(z_2\varsigma\beta + z_1(\varsigma' - v\cos\varphi))^2\end{array}}}{\Big((\varsigma\beta)^2 + (\varsigma' - v\cos\varphi)^2 + (v\varsigma\varphi' + v\sin\varphi)^2\Big)^{\frac{3}{2}}},\\
\tau_3 =& \frac{\begin{array}{l}(z_3(\varsigma' - v\cos\varphi) - z_2(v\varsigma\varphi' + v\sin\varphi))\big(z_1' - vz_2\beta\big)\\ +(z_3\varsigma\beta + z_1(v\varsigma\varphi' + v\sin\varphi))(z_2' + z_1 v\beta - z_3 v\varphi')\\ -(z_2\varsigma\beta + z_1(\varsigma' - v\cos\varphi))(z_3' + z_2 v\varphi')(y_3' + y_2 v\varphi')\end{array}}{\begin{array}{l}(z_3(\varsigma' - v\cos\varphi) - z_2(v\varsigma\varphi' + v\sin\varphi))^2 + (z_3\varsigma\beta + z_1(v\varsigma\varphi' + v\sin\varphi))^2\\ +(z_2\varsigma\beta + z_1(\varsigma' - v\cos\varphi))^2\end{array}}.
\end{aligned}
$$

Proof. We substitute the relations given in (56) and (57) in (2) and complete the proof. □

Remark 11. *The alternative frame for the alternative W-pedal curve α_W is established by the following equations:*

$$N_3 = N_3, \qquad C_3 = -\cos\psi T_3 + \sin\psi B_3, \qquad W_2 = \sin\psi T_3 + \cos\psi B_3, \tag{58}$$

where $\psi = \measuredangle(B_3, W_3)$, $\cos\psi = \dfrac{\kappa_3}{\sqrt{\kappa_3^2+\tau_3^2}}$, *and* $\sin\psi = \dfrac{\tau_3}{\sqrt{\kappa_3^2+\tau_3^2}}$.

Corollary 3. *The alternative frame formulae for the alternative W-pedal curve is given as follows:*

$$N_3' = \mu_3\beta_3 C_3, \qquad C_3' = -\mu_3\beta_3 N_3 + \mu_3\psi' W_3, \qquad {W_3}' = -\mu_3\psi' C_3, \tag{59}$$

where $\mu_3 = \|\alpha_W'\|$, $\psi' = \left(\dfrac{\tau_3}{\kappa_3}\right)'\left(\dfrac{\kappa_3^2+\tau_3^2}{\kappa_3^2}\right)$, $\beta_3 = \mu_3\sqrt{\kappa_3^2+\tau_3^2}$.

Definition 12. *The N_3C_3 Smarandache curve of the alternative W-pedal curve whose position vector is the linear combination of the two vectors N_3 and C_3 is defined by the following parametrization:*

$$\xi_1 = \frac{N_3 + C_3}{\sqrt{2}}. \tag{60}$$

Theorem 26. *Let $\{N_{\xi_1}, C_{\xi_1}, W_{\xi_1}\}$ denote the alternative frame for the N_3C_3 Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$\begin{aligned}
T_{\xi_1} =& \frac{\sqrt{2}(-\beta_3 N_3 + \beta_3 C_3 + \psi' W_3)}{\sqrt{2\beta_3^2 + {\psi'}^2}},\\
N_{\xi_1} =& B_{\xi_1} \wedge T_{\xi_1},\\
B_{\xi_1} =& \frac{(z_6\beta_3 - z_5\psi')N_3 + (z_6\beta_3 + z_4\psi')C_3 - (z_4\beta_3 + z_5\beta_3)W_3}{\sqrt{(z_6\beta_3 - z_5\psi')^2 + (z_6\beta_3 + z_4\psi')^2 + (z_4\beta_3 + z_5\beta_3)^2}},
\end{aligned}$$

where $z_4 = -\dfrac{(\mu_3\beta_3)' + \mu_3^2\beta_3^2}{\sqrt{2}}$, $z_5 = \dfrac{(\mu_3\beta_3)' - \mu_3^2\left(\beta_3^2 + {\psi'}^2\right)}{\sqrt{2}}$, $z_6 = \dfrac{(\mu_3\psi')' + \mu_3^2\beta_3\psi'}{\sqrt{2}}$.

Proof. If Relation (59) is taken into account, and the required derivatives for the N_3C_3 curve given in (60) are taken, we have

$$\begin{aligned}
\xi_1' =& \frac{-\mu_3\beta_3 N_2 + \mu_3\beta_3 C_2 + \mu_3\Delta' W_2}{\sqrt{2}},\\
\xi_1'' =& z_4 N_3 + z_5 C_3 + z_6 W_3,\\
\xi_1''' =& (z_4' - z_5\mu_3\beta_3)N_3 + (z_5' + z_4\mu_3\beta_3 - z_6\mu_3\psi')C_3 + (z_6' + z_5\mu_3\psi')W_3.
\end{aligned} \tag{61}$$

Moreover, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}
\xi_1' \wedge \xi_1'' =& \tfrac{\mu_3}{\sqrt{2}}((z_6\beta_3 - z_5\psi')N_3 + (z_6\beta_3 + z_4\psi')C_3 - \beta_3(z_4 + z_5)W_3),\\
\det(\xi_1', \xi_1'', \xi_1''') =& \tfrac{\mu_3}{\sqrt{2}}\begin{pmatrix} (z_4' - z_5\mu_3\beta_3)(z_6\beta_3 - z_5\psi') \\ +(z_5' + z_4\mu_3\beta_3 - z_6\mu_3\psi')(z_6\beta_3 + z_4\psi') \\ -\beta_3(z_4 + z_5)(z_6' + z_5\mu_3\psi') \end{pmatrix},
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
\|\xi_1'\| =& \tfrac{\mu_3}{\sqrt{2}}\sqrt{2\beta_3^2 + {\psi'}^2},\\
\|\xi_1' \wedge \xi_1''\| =& \tfrac{\mu_3}{\sqrt{2}}\sqrt{(z_6\beta_3 - z_5\psi')^2 + (z_6\beta_3 + z_4\psi')^2 + (z_4\beta_3 + z_5\beta_3)^2}.
\end{aligned} \tag{63}$$

Finally, by substituting Relations (61)–(63) into (1), we complete the proof. □

Theorem 27. *Let κ_{ξ_1} and τ_{ξ_1} be the curvature and the torsion functions for the N_3C_3 Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\kappa_{\xi_1} = \frac{2\sqrt{(z_6\beta_3 - z_5\psi')^2 + (z_6\beta_3 + z_4\psi')^2 + (z_4\beta_3 + z_5\beta_3)^2}}{\mu_3^2\left(2\beta_3^2 + \psi'^2\right)\sqrt{2\beta_3^2 + \psi'^2}},$$

$$\tau_{\xi_1} = \frac{\sqrt{2}\left(\begin{array}{l}(z_4' - z_5\mu_3\beta_3)(z_6\beta_3 - z_5\psi') + (z_5' + z_4\mu_3\beta_3 - z_6\mu_3\psi')(z_6\beta_3 + z_4\psi') \\ -\beta_3(z_4 + z_5)(z_6' + z_5\mu_3\psi')\end{array}\right)}{\mu_3\left((z_6\beta_3 - z_5\psi')^2 + (z_6\beta_3 + z_4\psi')^2 + (z_4\beta_3 + z_5\beta_3)^2\right)}.$$

Proof. By using (62) and (63) and substituting them into (2), we complete the proof. □

Remark 12. *The alternative frame for the N_3C_3 Smarandache curve ξ_1 is established by the following equations:*

$$N_{\xi_1} = N_{\xi_1}, \quad C_{\xi_1} = -\cos\Delta_1 T_{\xi_1} + \sin\Delta_1 B_{\xi_1}, \quad W_{\xi_1} = \sin\Delta_1 T_{\xi_1} + \cos\Delta_1 B_{\xi_1},$$

where $\Delta_1 = \measuredangle\left(B_{\xi_1}, W_{\xi_1}\right)$, $\cos\Delta_1 = \dfrac{\kappa_{\xi_1}}{\sqrt{\kappa_{\xi_1}^2 + \tau_{\xi_1}^2}}$, *and* $\sin\Delta_1 = \dfrac{\tau_{\xi_1}}{\sqrt{\kappa_{\xi_1}^2 + \tau_{\xi_1}^2}}$.

Definition 13. *The N_3W_3 Smarandache curve of the alternative W-pedal curve whose position vector is the linear combination of the two vectors N_3 and W_3 is defined by the following parametrization:*

$$\xi_2 = \frac{N_3 + W_3}{\sqrt{2}}. \tag{64}$$

Theorem 28. *Let $\{N_{\xi_2}, C_{\xi_2}, W_{\xi_2}\}$ denote the alternative frame for the N_3W_3 Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$T_{\xi_2} = C_2, \; N_{\xi_2} = \frac{-\beta_3 N_3 + \psi' W_3}{\sqrt{\psi'^2 + \beta_3^2}}, \; B_{\xi_2} = \frac{\psi' N_3 + \beta_3 W_3}{\sqrt{\psi'^2 + \beta_3^2}}.$$

Proof. If Relation (59) is taken into account, and the required derivatives for the N_3W_3 curve given in (64) are taken, we have

$$\begin{aligned}
\xi_2' =& \frac{\mu_3(\beta_3 - \psi')C_3}{\sqrt{2}}, \\
\xi_2'' =& \frac{-\mu_3^2\beta_3(\beta_3 - \psi')N_3 + (\mu_3(\beta_3 - \psi'))'C_3 + \mu_3^2\psi'(\beta_3 - \psi')W_3}{\sqrt{2}}, \\
\xi_2''' =& \frac{\begin{array}{l}-\left((\mu_3^2\beta_3(\beta_3 - \psi'))' + \mu_3\beta_3((\mu_3\beta_3 - \mu_3\psi'))'\right)N_3 \\ +\left(((\mu_3\beta_3 - \mu_3\psi'))'' - \mu_3^3\left(\beta_3^2 + \psi'^2\right)(\beta_3 - \psi')\right)C_3 \\ +\left(\mu_3\psi'(\mu_3(\beta_3 - \psi'))' + (\mu_3^2\psi'(\beta_3 - \psi'))'\right)W_3\end{array}}{\sqrt{2}}.
\end{aligned} \tag{65}$$

Moreover, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}
\xi_2' \wedge \xi_2'' =& \frac{\mu_3^3\psi'(\beta_3 - \psi')^2 N_3 + \mu_3^3\beta_3(\beta_3 - \psi')^2 W_3}{2}, \\
\det(\xi_2', \xi_2'', \xi_2''') =& \frac{\mu_3^5(\beta_3 - \psi')^3(\beta_3\psi'' - \psi'\beta_3')}{2\sqrt{2}},
\end{aligned} \tag{66}$$

and

$$\begin{aligned} \left\|\xi_2'\right\| =& \frac{\mu_3(\beta_3-\psi')}{\sqrt{2}}, \\ \left\|\xi_2' \wedge \xi_2''\right\| =& \frac{(\beta_3-\psi')^2\mu_3^3\sqrt{\psi'^2+\beta_3^2}}{2}. \end{aligned} \tag{67}$$

Finally, by substituting Relations (65)–(67) into (1), we complete the proof. □

Theorem 29. *Let κ_{ξ_2} and τ_{ξ_2} be the curvature and the torsion functions for the N_3W_3 Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\kappa_{\xi_2} = \frac{\sqrt{2}\sqrt{\Delta'^2+\beta_2^2}}{\beta_2-\Delta'}, \qquad \tau_{\xi_2} = \frac{\sqrt{2}(\beta_3\psi''-\psi'\beta_3')}{\mu_3(\beta_3-\psi')\left(\psi'^2+\beta_3^2\right)}.$$

Proof. By using (66) and (67) and substituting them into (2), we complete the proof. □

Remark 13. *The alternative frame for the N_3W_3 Smarandache curve ξ_2 is established by the following equations:*

$$N_{\xi_2} = N_{\xi_2}, \quad C_{\xi_2} = -\cos\Delta_2 T_{\xi_2} + \sin\Delta_2 B_{\xi_2}, \quad W_{\xi_2} = \sin\Delta_2 T_{\xi_2} + \cos\Delta_2 B_{\xi_2},$$

where $\Delta_2 = \measuredangle\left(B_{\xi_2}, W_{\xi_2}\right)$, $\cos\Delta_2 = \dfrac{\kappa_{\xi_2}}{\sqrt{\kappa_{\xi_2}^2+\tau_{\xi_2}^2}}$, *and* $\sin\Delta_2 = \dfrac{\tau_{\xi_2}}{\sqrt{\kappa_{\xi_2}^2+\tau_{\xi_2}^2}}$.

Definition 14. *The C_3W_3 Smarandache curve of the alternative W-pedal curve whose position vector is the linear combination of the two vectors C_3 and W_3 is defined by the following parametrization:*

$$\xi_3 = \frac{C_3+W_3}{\sqrt{2}}. \tag{68}$$

Theorem 30. *Let $\{N_{\xi_3}, C_{\xi_3}, W_{\xi_3}\}$ denote the alternative frame for the C_3W_3 Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$\begin{aligned} T_{\xi_3} =& \frac{-\beta_3N_3-\psi'C_3+\psi'W_3}{\sqrt{\beta_3^2+2\psi'^2}}, \\ N_{\xi_3} =& B_{\xi_3}\wedge T_{\xi_3}, \\ B_{\xi_3} =& \frac{-(\psi'z_9+\psi'z_8)N_3+(\beta_3z_9+\psi'z_7)C_3+(\psi'z_7-\beta_3z_8)W_3}{\sqrt{(\psi'z_9+\psi'z_8)^2+(\beta_3z_9+\psi'z_7)^2+(\psi'z_7-\beta_3z_8)^2}}, \end{aligned}$$

where $z_7 = \dfrac{\mu_3^2\psi'\beta_3-(\mu_3\beta_3)'}{\sqrt{2}}$, $z_8 = -\dfrac{(\mu_3\psi')'+\mu_3^2\left(\beta_3^2+\psi'^2\right)}{\sqrt{2}}$, $z_9 = \dfrac{(\mu_3\psi')'-\mu_3^2\psi'^2}{\sqrt{2}}$.

Proof. If Relation (59) is taken into account, and the required derivatives for the C_3W_3 curve given in (68) are taken, we have

$$\begin{aligned} \xi_3' =& \frac{-\mu_3\beta_3N_3-\mu_3\psi'C_3+\mu_3\psi'W_3}{\sqrt{2}}, \\ \xi_3'' =& z_7N_3+z_8C_3+z_9W_3, \\ \xi_3''' =& \left(z_7'-z_8\mu_3\beta_3\right)N_3+\left(z_8'+z_7\mu_3\beta_3-z_9\mu_3\psi'\right)C_3+\left(z_9'+z_8\mu_3\psi'\right)W_3. \end{aligned} \tag{69}$$

Moreover, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}\xi_3' \wedge \xi_3'' =& \frac{\mu_3(-\psi'(z_9+z_8)N_3+(\beta_3 z_9+\psi' z_7)C_3+(\psi' z_7-\beta_3 z_8)W_3)}{\sqrt{2}},\\ \det(\xi_3', \xi_3'', \xi_3''') =& \frac{\mu_3}{\sqrt{2}}\begin{pmatrix} (\psi' z_7-\beta_3 z_8)(z_9'+z_8\mu_3\Delta')-\psi'(z_9+z_8)(z_7'-z_8\mu_3\beta_3)\\ +(\beta_3 z_9+\psi' z_7)(z_8'+z_7\mu_3\beta_3-z_9\mu_3\psi')\end{pmatrix},\end{aligned} \tag{70}$$

and

$$\begin{aligned}\|\xi_3'\| =& \frac{\mu_3}{\sqrt{2}}\sqrt{\beta_3^2+2\psi'^2},\\ \|\xi_3' \wedge \xi_3''\| =& \frac{\mu_3}{\sqrt{2}}\sqrt{(\psi' x_9+\psi' x_8)^2+(\beta_3 z_9+\psi' z_7)^2+(\psi' z_7-\beta_3 z_8)^2}.\end{aligned} \tag{71}$$

Finally, by substituting Relations (69)–(71) into (1), we complete the proof. □

Theorem 31. *Let κ_{ξ_3} and τ_{ξ_3} be the curvature and the torsion functions for the C_3W_3 Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\begin{aligned}\kappa_{\xi_3} =& \frac{2\sqrt{(\psi' z_9+\psi' z_8)^2+(\beta_3 z_9+\psi' z_7)^2+(\psi' z_7-\beta_3 z_8)^2}}{\mu_3^2\left(\beta_3^2+2\psi'^2\right)\sqrt{\beta_3^2+2\psi'^2}},\\ \tau_{\xi_3} =& \frac{\sqrt{2}\begin{pmatrix} (\psi' z_7-\beta_3 z_8)(z_9'+z_8\mu_3\Delta')-\psi'(z_9+z_8)(z_7'-z_8\mu_3\beta_3)\\ +(\beta_3 z_9+\psi' z_7)(z_8'+z_7\mu_3\beta_3-z_9\mu_3\psi')\end{pmatrix}}{\mu_3\left((\psi' z_9+\psi' z_8)^2+(\beta_3 z_9+\psi' z_7)^2+(\psi' z_7-\beta_3 z_8)^2\right)}.\end{aligned}$$

Proof. By using (70) and (71) and substituting them into (2), we complete the proof. □

Remark 14. *The alternative frame for the C_3W_3 Smarandache curve ξ_3 is established by the following equations:*

$$N_{\xi_3} = N_{\xi_3}, \quad C_{\xi_3} = -\cos\Delta_3 T_{\xi_3}+\sin\Delta_3 B_{\xi_3}, \quad W_{\xi_3} = \sin\Delta_3 T_{\xi_3}+\cos\Delta_3 B_{\xi_3},$$

where $\Delta_3 = \measuredangle(B_{\xi_3}, W_{\xi_3})$, $\cos\Delta_3 = \dfrac{\kappa_{\xi_3}}{\sqrt{\kappa_{\xi_3}^2+\tau_{\xi_3}^2}}$, *and* $\sin\Delta_3 = \dfrac{\tau_{\xi_3}}{\sqrt{\kappa_{\xi_3}^2+\tau_{\xi_3}^2}}$.

Definition 15. *The $N_3C_3W_3$ Smarandache curve of the alternative W-pedal curve whose position vector is the linear combination of the vectors N_3, C_3 and W_3 is defined by the following parametrization:*

$$\xi_4 = \frac{N_3+C_3+W_3}{\sqrt{2}}. \tag{72}$$

Theorem 32. *Let $\{N_{\xi_4}, C_{\xi_4}, W_{\xi_4}\}$ denote the alternative frame for the $N_3C_3W_3$ Smarandache curve. Then, we have the following relations among the Frenet vectors:*

$$\begin{aligned}T_{\xi_4} =& \frac{-\beta_3 N_3+(\beta_3-\psi')C_3+\psi' W_3}{\sqrt{2}\sqrt{\beta_3^2-\beta_3\psi'+\psi'^2}},\\ N_{\xi_4} =& B_{\xi_4}\wedge T_{\xi_4},\\ B_{\xi_4} =& \frac{-(z_{11}\psi'-z_{12}(\beta_3-\psi'))N_3+(z_{10}\psi'+z_{12}\beta_3)C_3-(z_{11}\beta_3+z_{10}(\beta_3-\psi'))W_2}{\sqrt{(z_{11}\psi'-z_{12}(\beta_3-\psi'))^2+(z_{10}\psi'+z_{12}\beta_3)^2+(z_{11}\beta_3+z_{10}(\beta_3-\psi'))^2}},\end{aligned}$$

$$\text{where } z_{10} = \frac{(\mu_3\beta_3)' + \mu_3^2\beta_3(\beta_3 - \psi')}{\sqrt{3}}, \quad z_{11} = \frac{(\mu_3(\beta_3 - \psi'))' - \mu_3^2\left(\beta_3^2 + \psi'^2\right)}{\sqrt{3}},$$
$$z_{12} = \frac{(\mu_3\psi')' + \mu_3^2\psi'(\beta_3 - \psi')}{\sqrt{3}}.$$

Proof. If Relation (59) is taken into account, and the required derivatives for the $N_3C_3W_3$ curve given in (72) are taken, we have

$$\begin{aligned}\xi_4' &= \frac{\mu_3(-\beta_3 N_3 + (\beta_3 - \psi')C_3 + \psi' W_3)}{\sqrt{3}},\\ \xi_4'' &= z_{10}N_3 + z_{11}C_3 + z_{12}W_3,\\ \xi_4''' &= (z_{10} - z_{11}\mu_3\beta_3)N_3 + \left(z_{11} + z_{10}\mu_3\beta_3 - z_{12}\mu_3\psi'\right)C_3 + \left(z_{12} + z_{11}\mu_3\psi'\right)W_3.\end{aligned} \tag{73}$$

Moreover, if the vector products, determinants and norms are taken, we obtain:

$$\begin{aligned}\xi_4' \wedge \xi_4'' &= \frac{\mu_3\begin{pmatrix} -(z_{11}\psi' - z_{12}(\beta_2 - \psi'))N_3 + (z_{10}\psi' + z_{12}\beta_3)C_3 \\ -(z_{11}\beta_3 + z_{10}(\beta_3 - \psi'))W_3 \end{pmatrix}}{\sqrt{3}},\\ \det(\xi_4', \xi_4'', \xi_4''') &= \frac{\mu_3}{\sqrt{3}}\begin{pmatrix} -(z_{11}\psi' - z_{12}(\beta_3 - \psi'))(z_{10} - z_{11}\mu_3\beta_3) \\ +(z_{10}\psi' + z_{12}\beta_3)(z_{11} + z_{10}\mu_3\beta_3 - z_{12}\mu_3\psi') \\ -(z_{11}\beta_3 + z_{10}(\beta_3 - \psi'))(z_{12} + z_{11}\mu_3\psi') \end{pmatrix},\end{aligned} \tag{74}$$

and

$$\begin{aligned}\|\xi_4'\| &= \frac{\sqrt{6}\mu_3}{3}\sqrt{\beta_3^2 - \beta_3\psi' + \psi'^2},\\ \|\xi_4' \wedge \xi_4''\| &= \frac{\mu_3}{\sqrt{3}}\sqrt{(z_{11}\psi' - z_{12}(\beta_2 - \psi'))^2 + (z_{10}\psi' + z_{12}\beta_3)^2 + (z_{11}\beta_3 + z_{10}(\beta_3 - \psi'))^2}.\end{aligned} \tag{75}$$

Finally, by substituting Relations (73)–(75) into (1), we complete the proof. □

Theorem 33. *Let κ_{ξ_4} and τ_{ξ_4} be the curvature and the torsion functions for the $N_3C_3W_3$ Smarandache curve, respectively. Then, we have the following relations among the curvatures:*

$$\kappa_{\xi_4} = \frac{3\sqrt{2}}{4} \cdot \frac{\sqrt{(z_{11}\psi' - z_{12}(\beta_2 - \psi'))^2 + (z_{10}\psi' + z_{12}\beta_3)^2 + (z_{11}\beta_3 + z_{10}(\beta_3 - \psi'))^2}}{\mu_3^2\left(\beta_3^2 - \beta_3\psi' + \psi'^2\right)\sqrt{\beta_3^2 - \beta_3\psi' + \psi'^2}},$$
$$\tau_{\xi_4} = \frac{\sqrt{3}\begin{pmatrix} -(z_{11}\psi' - z_{12}(\beta_3 - \psi'))(z_{10} - z_{11}\mu_3\beta_3) + (z_{10}\psi' + z_{12}\beta_3)(z_{11} + z_{10}\mu_3\beta_3 - z_{12}\mu_3\psi') \\ -(z_{11}\beta_3 + z_{10}(\beta_3 - \psi'))(z_{12} + z_{11}\mu_3\psi') \end{pmatrix}}{\mu_3\left((z_{11}\psi' - z_{12}(\beta_2 - \psi'))^2 + (z_{10}\psi' + z_{12}\beta_3)^2 + (z_{11}\beta_3 + z_{10}(\beta_3 - \psi'))^2\right)}.$$

Proof. By using (74) and (75) and substituting them into (2), we complete the proof. □

Remark 15. *The alternative frame for the $N_3C_3W_3$ Smarandache curve ξ_4 is established by the following equations:*

$$N_{\xi_4} = N_{\xi_4}, \quad C_{\xi_4} = -\cos\Delta_4 T_{\xi_4} + \sin\Delta_4 B_{\xi_4}, \quad W_{\xi_4} = \sin\Delta_4 T_{\xi_4} + \cos\Delta_4 B_{\xi_4},$$

where $\Delta_4 = \measuredangle(B_{\xi_4}, W_{\xi_4})$, $\cos\Delta_4 = \dfrac{\kappa_{\xi_4}}{\sqrt{\kappa_{\xi_4}^2 + \tau_{\xi_4}^2}}$, *and* $\sin\Delta_4 = \dfrac{\tau_{\xi_4}}{\sqrt{\kappa_{\xi_4}^2 + \tau_{\xi_4}^2}}$.

By using Example 1, Smarandache curves of the alternative W-pedal curve according to the point $P(1,1,1)$ are illustrated in Figure 7.

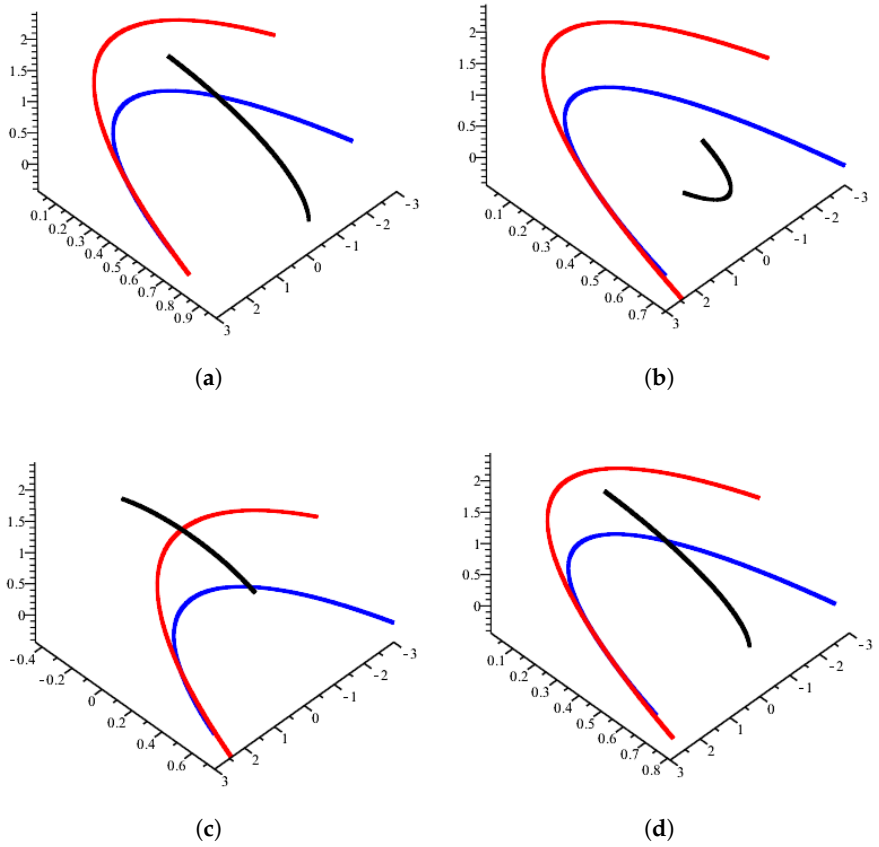

Figure 7. Smarandache curves (black) of the alternative W-pedal curve (red) of the curve $\zeta(t)$ (blue) according to the fixed point $P(1,1,1)$ where $t \in [-0.5, 0.5]$. (**a**) N_3C_3 Smarandache curve; (**b**) N_3W_3 Smarandache curve; (**c**) C_3W_3 Smarandache curve; (**d**) $N_3C_3W_3$ Smarandache curve.

5. Conclusions

The theory of curves is the most fundamental concept in differential geometry. Therefore, generating new curves is important as well as providing their characteristics mostly by moving frames along the curve and by its invariant curvatures. Pedal curves and Smarandache curves are two ways of defining new curves. In this study, we defined pedal-type curves by using the moving alternative frame vectors along the curve. Second, we established the relations among the alternative frame apparatus for the main curve and its pedal curve. Next, we introduced the Smarandache curves of the new pedal curves. Thus, we examined twelve new curves for one fixed point. A set of new curves could be investigated to expand the literature on the theory of curves by having different points and more curves added to the field. Finally, since symmetry as a concept is more relevant for real-world applications, with this paper, it is believed that the obtained formulas and definitions for the new curves can be useful for some design and modeling purposes. We also encourage researchers to examine possible pedal-like surfaces from this study.

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